International Tables for Crystallography (2006). Vol. A1, Section 1.2.6, pp. 16–18.

does not classify the infinite set of all lattices into a finite number of lattice types, because the same lattice may belong to space groups of different crystal classes. For example, the same lattice may be that of a space group of type $P1$ (of crystal class 1) and that of a space group of type $P\overline{1}$ (of crystal class $\overline{1}$).

Nevertheless, there is also a definition of the 'point group of a lattice'. Let a vector lattice $\mathbf{L}$ of a space group $\mathcal{G}$ be referred to a lattice basis. Then the linear parts $\boldsymbol{W}$ of the matrix–column pairs $(\boldsymbol{W}, \boldsymbol{w})$ of $\mathcal{G}$ form the point group $\mathcal{P}_{\mathcal{G}}$. If $(\boldsymbol{W}, \boldsymbol{w})$ maps the space group $\mathcal{G}$ onto itself, then the linear part $\boldsymbol{W}$ maps the (vector) lattice $\mathbf{L}$ onto itself. However, there may be additional matrices which also describe symmetry operations of the lattice $\mathbf{L}$. For example, the point group $\mathcal{P}_{\mathcal{G}}$ of a space group of type $P1$ consists of the identity $\boldsymbol{I}$ only. However, with any vector $\mathbf{t} \in \mathbf{L}$, the negative vector $-\mathbf{t} \in \mathbf{L}$ also belongs to $\mathbf{L}$. Therefore, the lattice $\mathbf{L}$ is always centrosymmetric and has the inversion $\overline{\boldsymbol{I}}$ as a symmetry operation independent of the symmetry of the space group.

Definition 1.2.5.4.4. The set of all orthogonal mappings with matrices $\boldsymbol{W}$ which map a lattice $\mathbf{L}$ onto itself is called the point group of the lattice $\mathbf{L}$ or the *holohedry* of the lattice $\mathbf{L}$. A crystal class of point groups $\mathcal{P}_{\mathcal{G}}$ is called a *holohedral crystal class* if it contains a holohedry. □

There are seven holohedral crystal classes in the space: $\overline{1}$, $2/m$, mmm, $4/mmm$, $\overline{3}m$, $6/mmm$ and $m\overline{3}m$. Their lattices are called triclinic, monoclinic, orthorhombic, tetragonal, rhombohedral, hexagonal and cubic, respectively. There are four holohedral crystal classes in the plane: 2, $2mm$, $4mm$ and $6mm$. Their two-dimensional lattices (or nets) are called oblique, rectangular, square and hexagonal, respectively.

The lattices can be classified into *lattice types* or *Bravais types*, mostly called *Bravais lattices*, or into *lattice systems* (called *Bravais systems* in editions 1 to 4 of *IT* A). These classifications are not discussed here because they are not directly relevant to the classification of the space groups. This is because the lattice symmetry is not necessarily typical for the symmetry of its space group but may accidentally be higher. For example, the lattice of a monoclinic crystal may be accidentally orthorhombic (only for certain values of temperature and pressure). In Sections 8.2.5 and 8.2.7 of *IT* A the 'typical lattice symmetry' of a space group is defined.

1.2.5.5. *Crystal systems and crystal families*

The example of $P1$ mentioned above shows that the point group of the lattice may be systematically of higher order than that of its space group. There are obviously point groups and thus space groups that belong to a holohedral crystal class and those that do not. The latter can be assigned to a holohedral crystal class uniquely according to the following definition:[8]

Definition 1.2.5.5.1. A crystal class $\mathbf{C}$ of a space group $\mathcal{G}$ is either holohedral $\mathbf{H}$ or it can be assigned uniquely to $\mathbf{H}$ by the condition: any point group of $\mathbf{C}$ is a subgroup of a point group of $\mathbf{H}$ but not a subgroup of a holohedral crystal class $\mathbf{H}'$ of smaller order. The set of all crystal classes of space groups that are assigned to the same holohedral crystal class is called a *crystal system* of space groups. □

The 32 crystal classes of space groups are classified into seven crystal systems which are called *triclinic, monoclinic, orthorhombic, tetragonal, trigonal, hexagonal and cubic*. There are four crystal systems of plane groups: *oblique, rectangular, square and hexagonal*. Like the space groups, the crystal classes of point groups are classified into the seven crystal systems of point groups.

Apart from accidental lattice symmetries, the space groups of different crystal systems have lattices of different symmetry. As an exception, the hexagonal primitive lattice occurs in both hexagonal and trigonal space groups as the typical lattice. Therefore, the space groups of the trigonal and the hexagonal crystal systems are more related than space groups from other different crystal systems. Indeed, in different crystallographic schools the term 'crystal system' was used for different objects. One sense of the term was the 'crystal system' as defined above, while another sense of the old term 'crystal system' is now called a 'crystal family' according to the following definition [for definitions that are also valid in higher-dimensional spaces, see Brown *et al.* (1978) or *IT* A, Chapter 8.2]:

Definition 1.2.5.5.2. In three-dimensional space, the classification of the set of all space groups into crystal families is the same as that into crystal systems with the one exception that the trigonal and hexagonal crystal systems are united to form the *hexagonal crystal family*. There is no difference between crystal systems and crystal families in the plane. □

The partition of the space groups into crystal families is the most universal one. The space groups and their types, their crystal classes and their crystal systems are classified by the crystal families. Analogously, the crystallographic point groups and their crystal classes and crystal systems are classified by the crystal families of point groups. Lattices, their Bravais types and lattice systems can also be classified into crystal families of lattices; *cf. IT* A, Chapter 8.2.

1.2.6. Types of subgroups of space groups

1.2.6.1. *Introductory remarks*

Group–subgroup relations form an essential part of the applications of space-group theory. Let $\mathcal{G}$ be a space group and $\mathcal{H} < \mathcal{G}$ a proper subgroup of $\mathcal{G}$. All maximal subgroups $\mathcal{H} < \mathcal{G}$ of any space group $\mathcal{G}$ are listed in Part 2 of this volume. There are different kinds of subgroups which are defined and described in this section. The tables and graphs of this volume are arranged according to these kinds of subgroups. Moreover, for the different kinds of subgroups different data are listed in the subgroup tables and graphs.

Let $\mathcal{G}_j$ and $\mathcal{H}_j$ be space groups of the space-group types $\mathcal{G}$ and $\mathcal{H}$. The group–subgroup relation $\mathcal{G}_j > \mathcal{H}_j$ is a relation between the particular space groups $\mathcal{G}_j$ and $\mathcal{H}_j$ but it can be generalized to the space-group types $\mathcal{G}$ and $\mathcal{H}$. Certainly, not every space group of the type $\mathcal{H}$ will be a subgroup of every space group of the type $\mathcal{G}$. Nevertheless, the relation $\mathcal{G}_j > \mathcal{H}_j$ holds for any space group of $\mathcal{G}$ and $\mathcal{H}$ in the following sense: If $\mathcal{G}_j > \mathcal{H}_j$ holds for the pair $\mathcal{G}_j$ and $\mathcal{H}_j$, then for any space group $\mathcal{G}_k$ of the type $\mathcal{G}$ a space group $\mathcal{H}_k$ of the type $\mathcal{H}$ exists for which the corresponding relation $\mathcal{G}_k > \mathcal{H}_k$ holds. Conversely, for any space group $\mathcal{H}_m$ of the type $\mathcal{H}$ a space group $\mathcal{G}_m$ of the type $\mathcal{G}$ exists for which the corresponding relation $\mathcal{G}_m > \mathcal{H}_m$ holds. Only this property of the group–subgroup relations made it possible to compile and arrange the tables of this volume so that they are as concise as those of *IT* A.

1.2.6.2. *Definitions and examples*

'Maximal subgroups' have been introduced by Definition 1.2.4.1.2. The importance of this definition will become apparent

[8] This assignment does hold for low dimensions of space at least up to dimension 4. A dimension-independent definition of the concepts of 'crystal system' and 'crystal family' is found in *IT* A, Chapter 8.2, where the classifications are treated in more detail.

in the corollary to Hermann's theorem, *cf.* Lemma 1.2.8.1.3. In this volume only the maximal subgroups are listed for any plane and any space group. A maximal subgroup of a plane group is a plane group, a maximal subgroup of a space group is a space group. On the other hand, a minimal supergroup of a plane group or of a space group is not necessarily a plane group or a space group, *cf.* Section 2.1.6.

If the maximal subgroups are known for each space group, then each non-maximal subgroup of a space group $\mathcal{G}$ with finite index can in principle be obtained from the data on maximal subgroups. A non-maximal subgroup $\mathcal{H} < \mathcal{G}$ of finite index $[i]$ is connected with the original group $\mathcal{G}$ through a chain $\mathcal{H} = \mathcal{Z}_k < \mathcal{Z}_{k-1} < \ldots < \mathcal{Z}_1 < \mathcal{Z}_0 = \mathcal{G}$, where each group $\mathcal{Z}_j < \mathcal{Z}_{j-1}$ is a maximal subgroup of $\mathcal{Z}_{j-1}$, with the index $[i_j] = |\mathcal{Z}_{j-1} : \mathcal{Z}_j|$, $j = 1, \ldots, k$. The number k is finite and the relation $i = \prod_{j=1}^{k} i_j$ holds, *i.e.* the total index $[i]$ is the product of the indices i_j.

According to Hermann (1929), the following types of subgroups of space groups have to be distinguished:

Definition 1.2.6.2.1. A subgroup $\mathcal{H}$ of a space group $\mathcal{G}$ is called a *translationengleiche subgroup* or a *t*-subgroup of $\mathcal{G}$ if the set $\mathcal{T}(\mathcal{G})$ of translations is retained, *i.e.* $\mathcal{T}(\mathcal{H}) = \mathcal{T}(\mathcal{G})$, but the number of cosets of $\mathcal{G}/\mathcal{T}(\mathcal{G})$, *i.e.* the order P of the point group $\mathcal{P}_{\mathcal{G}}$, is reduced such that $|\mathcal{G}/\mathcal{T}(\mathcal{G})| > |\mathcal{H}/\mathcal{T}(\mathcal{H})|$.[9] □

The order of a crystallographic point group $\mathcal{P}_{\mathcal{G}}$ of the space group $\mathcal{G}$ is always finite. Therefore, the number of the subgroups of $\mathcal{P}_{\mathcal{G}}$ is also always finite and these subgroups and their relations are displayed in well known graphs, *cf.* Chapter 2.4 and Section 2.1.7 of this volume. Because of the isomorphism between the point group $\mathcal{P}_{\mathcal{G}}$ and the factor group $\mathcal{G}/\mathcal{T}(\mathcal{G})$, the subgroup graph for the point group $\mathcal{P}_{\mathcal{G}}$ is the same as that for the *t*-subgroups of $\mathcal{G}$, only the labels of the groups are different. For deviations between the point-group graphs and the actual space-group graphs of Chapter 2.4, *cf.* Section 2.1.7.2.

Example 1.2.6.2.2.

Consider a space group $\mathcal{G}$ of type $P12/m1$ referred to a conventional coordinate system. The translation subgroup $\mathcal{T}(\mathcal{G})$ consists of all translations with translation vectors $\mathbf{t} = u\mathbf{a}+v\mathbf{b}+w\mathbf{c}$, where u, v, w run through all integer numbers. The coset decomposition of $(\mathcal{G} : \mathcal{T}(\mathcal{G}))$ results in the four cosets $\mathcal{T}(\mathcal{G})$, $\mathcal{T}(\mathcal{G})\,\boldsymbol{2}_0$, $\mathcal{T}(\mathcal{G})\,\boldsymbol{m}_0$ and $\mathcal{T}(\mathcal{G})\,\overline{\boldsymbol{1}}_0$, where the right operations are a twofold rotation $\boldsymbol{2}_0$ around the rotation axis passing through the origin, a reflection $\boldsymbol{m}_0$ through a plane containing the origin and an inversion $\overline{\boldsymbol{1}}_0$ with the inversion point at the origin, respectively. The three combinations $\mathcal{H}_1 = \mathcal{T}(\mathcal{G}) \cup \mathcal{T}(\mathcal{G})\,\boldsymbol{2}_0$, $\mathcal{H}_2 = \mathcal{T}(\mathcal{G}) \cup \mathcal{T}(\mathcal{G})\,\boldsymbol{m}_0$ and $\mathcal{H}_3 = \mathcal{T}(\mathcal{G}) \cup (\mathcal{T}\mathcal{G})\,\overline{\boldsymbol{1}}_0$ each form a *translationengleiche* maximal subgroup of $\mathcal{G}$ of index 2 with the space-group symbols $P121$, $P1m1$ and $P\overline{1}$, respectively.

Definition 1.2.6.2.3. A subgroup $\mathcal{H} < \mathcal{G}$ of a space group $\mathcal{G}$ is called a *klassengleiche subgroup* or a *k-subgroup* if the set $\mathcal{T}(\mathcal{G})$ of all translations of $\mathcal{G}$ is reduced to $\mathcal{T}(\mathcal{H}) < \mathcal{T}(\mathcal{G})$ but all linear parts of $\mathcal{G}$ are retained. Then the number of cosets of the decompositions $\mathcal{H}/\mathcal{T}(\mathcal{H})$ and $\mathcal{G}/\mathcal{T}(\mathcal{G})$ is the same, *i.e.* $|\mathcal{H}/\mathcal{T}(\mathcal{H})| = |\mathcal{G}/\mathcal{T}(\mathcal{G})|$. In other words: the order of the point group $\mathcal{P}_{\mathcal{H}}$ is the same as that of $\mathcal{P}_{\mathcal{G}}$. See also footnote 9. □

For a *klassengleiche* subgroup $\mathcal{H} < \mathcal{G}$, the cosets of the factor group $\mathcal{H}/\mathcal{T}(\mathcal{H})$ are smaller than those of $\mathcal{G}/\mathcal{T}(\mathcal{G})$. Because $\mathcal{T}(\mathcal{H})$ is still infinite, the number of elements of each coset is infinite but the index $|\mathcal{T}(\mathcal{G}) : \mathcal{T}(\mathcal{H})| > 1$ is finite. The number of *k*-subgroups of $\mathcal{G}$ is always infinite.

Example 1.2.6.2.4.

Consider a space group $\mathcal{G}$ of the type $C121$, referred to a conventional coordinate system. The set $\mathcal{T}(\mathcal{G})$ of all translations can be split into the set $\mathcal{T}_i$ of all translations with integer coefficients u, v and w and the set $\mathcal{T}_f$ of all translations for which the coefficients u and v are fractional. The set $\mathcal{T}_i$ forms a group; the set $\mathcal{T}_f$ is the other coset in the decomposition $(\mathcal{T}(\mathcal{G}) : \mathcal{T}_i)$ and does not form a group. Let t_C be the 'centring translation' with the translation vector $\frac{1}{2}(\mathbf{a}+\mathbf{b})$. Then $\mathcal{T}_f$ can be written $\mathcal{T}_i\,t_C$. Let $\boldsymbol{2}_0$ mean a twofold rotation around the rotation axis through the origin. There are altogether four cosets of the decomposition $(\mathcal{G} : \mathcal{T}_i)$, which can be written now as $\mathcal{T}_i$, $\mathcal{T}_f = \mathcal{T}_i\,t_C$, $\mathcal{T}_i\,\boldsymbol{2}_0$ and $\mathcal{T}_f\,\boldsymbol{2}_0 = (\mathcal{T}_i\,t_C)\,\boldsymbol{2}_0 = \mathcal{T}_i\,(t_C\,\boldsymbol{2}_0)$. The union $\mathcal{T}_i \cup (\mathcal{T}_i\,t_C) = \mathcal{T}_{\mathcal{G}}$ forms the *translationengleiche* maximal subgroup $C1$ (conventional setting $P1$) of $\mathcal{G}$ of index 2. The union $\mathcal{T}_i \cup (\mathcal{T}_i\,\boldsymbol{2}_0)$ forms the *klassengleiche* maximal subgroup $P121$ of $\mathcal{G}$ of index 2. The union $\mathcal{T}_i \cup (\mathcal{T}_i(t_C\,\boldsymbol{2}_0))$ also forms a *klassengleiche* maximal subgroup of index 2. Its HM symbol is $P12_11$, and the twofold rotations $\boldsymbol{2}$ of the point group $\boldsymbol{2}$ are realized by screw rotations $\boldsymbol{2}_1$ in this subgroup because $(t_C\,\boldsymbol{2}_0)$ is a screw rotation with its screw axis running parallel to the $\mathbf{b}$ axis through the point $\frac{1}{4}, 0, 0$. There are in fact these two *k*-subgroups of $C121$ of index 2 which have the group $\mathcal{T}_i$ in common. In the subgroup table of $C121$ both are listed under the heading 'Loss of centring translations' because the conventional unit cell is retained while only the centring translations have disappeared. (Four additional *klassengleiche* maximal subgroups of $C121$ are found under the heading 'Enlarged unit cell'.)

The group $\mathcal{T}_i$ of type $P1$ is a non-maximal subgroup of $C121$ of index 4.

Definition 1.2.6.2.5. A *klassengleiche* or *k*-subgroup $\mathcal{H} < \mathcal{G}$ is called *isomorphic* or an *isomorphic subgroup* if it belongs to the same affine space-group type (isomorphism type) as $\mathcal{G}$. If a subgroup is not isomorphic, it is sometimes called *non-isomorphic*. □

Isomorphic subgroups are special *k*-subgroups. The importance of the distinction between *k*-subgroups in general and isomorphic subgroups in particular stems from the fact that the number of maximal non-isomorphic *k*-subgroups of any space group is finite, whereas the number of maximal isomorphic subgroups is always infinite, see Section 1.2.8.

Example 1.2.6.2.6.

Consider a space group $\mathcal{G}$ of type $P\overline{1}$ referred to a conventional coordinate system. The translation subgroup $\mathcal{T}(\mathcal{G})$ consists of all translations with translation vectors $\mathbf{t} = u\mathbf{a}+v\mathbf{b}+w\mathbf{c}$, where u, v and w run through all integer numbers. There is an inversion $\overline{\boldsymbol{1}}_0$ with the inversion point at the origin and also an infinite number of other inversions, generated by the combinations of $\overline{\boldsymbol{1}}_0$ with all translations of $\mathcal{T}(\mathcal{G})$.

We consider the subgroup $\mathcal{T}_g$ of all translations with an even coefficient u and arbitrary integers v and w as well as the coset decomposition $(\mathcal{G} : \mathcal{T}_g)$. Let t_a be the translation with the translation vector $\mathbf{a}$. There are four cosets: $\mathcal{T}_g$, $\mathcal{T}_g\,t_a$, $\mathcal{T}_g\,\overline{\boldsymbol{1}}_0$ and $\mathcal{T}_g(t_a\,\overline{\boldsymbol{1}}_0)$. The union $\mathcal{T}_g \cup (\mathcal{T}_g\,t_a)$ forms the *translationengleiche* maximal subgroup $\mathcal{T}(\mathcal{G})$ of index 2. The union $\mathcal{T}_g \cup (\mathcal{T}_g\,\overline{\boldsymbol{1}}_0)$ forms an isomorphic maximal subgroup of index 2, as does the

[9] German: *zellengleiche* means 'with the same cell'; *translationengleiche* means 'with the same translations'; *klassengleiche* means 'of the same (crystal) class'. Of the different German declension endings only the form with terminal *-e* is used in this volume. The terms *zellengleiche* and *klassengleiche* were introduced by Hermann (1929). The term *zellengleiche* was later replaced by *translationengleiche* because of possible misinterpretations. In this volume they are sometimes abbreviated as *t*-subgroups and *k*-subgroups.

union $\mathcal{T}_g \cup (\mathcal{T}_g\,(t_a\,\overline{\mathbf{1}}_0))$. There are thus two maximal isomorphic subgroups of index 2 which are obtained by doubling the a lattice parameter. There are altogether 14 isomorphic subgroups of index 2 for any space group of type $P\,\overline{1}$ which are obtained by seven different cell enlargements.

If $\mathcal{G}$ belongs to a pair of enantiomorphic space-group types, then the isomorphic subgroups of $\mathcal{G}$ may belong to different crystallographic space-group types with different HM symbols and different space-group numbers. In this case, an infinite number of subgroups belong to the crystallographic space-group type of $\mathcal{G}$ and another infinite number belong to the enantiomorphic space-group type.

Example 1.2.6.2.7.

Space group $P4_1$, No. 76, has for any prime number $p > 2$ an isomorphic maximal subgroup of index p with the lattice parameters $a,\ b,\ pc$. This is an infinite number of subgroups because there is an infinite number of primes. The subgroups belong to the space-group type $P4_1$ if $p \equiv 1$ mod 4; they belong to the type $P4_3$ if $p \equiv 3$ mod 4.

Definition 1.2.6.2.8. A subgroup of a space group is called *general* or a *general subgroup* if it is neither a *translationengleiche* nor a *klassengleiche* subgroup. It has lost translations as well as linear parts, *i.e.* point-group symmetry. □

Example 1.2.6.2.9.

The subgroup $\mathcal{T}_g$ in Example 1.2.6.2.6 has lost all inversions of the original space group $P\overline{1}$ as well as all translations with odd u. It is a general subgroup $P1$ of the space group $P\overline{1}$ of index 4.

1.2.6.3. *The role of normalizers for group–subgroup pairs of space groups*

In Section 1.2.4.5, the normalizer $\mathcal{N}_{\mathcal{G}}(\mathcal{H})$ of a subgroup $\mathcal{H} < \mathcal{G}$ in the group $\mathcal{G}$ was defined. The equation $\mathcal{H} \trianglelefteq \mathcal{N}_{\mathcal{G}}(\mathcal{H}) \leq \mathcal{G}$ holds, *i.e.* $\mathcal{H}$ is a normal subgroup of $\mathcal{N}_{\mathcal{G}}(\mathcal{H})$. The normalizer $\mathcal{N}_{\mathcal{G}}(\mathcal{H})$, by its index in $\mathcal{G}$, determines the number $N_j = |\mathcal{G} : \mathcal{N}_{\mathcal{G}}(\mathcal{H})|$ of subgroups $\mathcal{H}_j < \mathcal{G}$ that are conjugate in the group $\mathcal{G}$, *cf.* Remarks (2) and (3) below Definition 1.2.4.5.1.

The group–subgroup relations between space groups become more transparent if one looks at them from a more general point of view. Space groups are part of the general theory of mappings. Particular groups are the *affine group* $\mathcal{A}$ of all reversible affine mappings, the *Euclidean group* $\mathcal{E}$ of all isometries, the *translation group* $\mathcal{T}$ of all translations and the *orthogonal group* $\mathcal{O}$ of all orthogonal mappings.

Connected with any particular space group $\mathcal{G}$ are its group of translations $\mathcal{T}(\mathcal{G})$ and its point group $\mathcal{P}_{\mathcal{G}}$. In addition, the normalizers $\mathcal{N}_{\mathcal{A}}(\mathcal{G})$ of $\mathcal{G}$ in the affine group $\mathcal{A}$ and $\mathcal{N}_{\mathcal{E}}(\mathcal{G})$ in the Euclidean group $\mathcal{E}$ are useful. They are listed in Section 15.2.1 of *IT* A. Although consisting of isometries only, $\mathcal{N}_{\mathcal{E}}(\mathcal{G})$ is not necessarily a space group, see the paragraph below Lemma 1.2.7.2.6.

For the group–subgroup pairs $\mathcal{H} < \mathcal{G}$ the following relations hold:

(1) $\mathcal{T}(\mathcal{H}) \leq \mathcal{H} \leq \mathcal{N}_{\mathcal{G}}(\mathcal{H}) \leq \mathcal{G} \leq \mathcal{N}_{\mathcal{E}}(\mathcal{G}) < \mathcal{E}$;

(1*a*) $\mathcal{H} \leq \mathcal{N}_{\mathcal{G}}(\mathcal{H}) \leq \mathcal{N}_{\mathcal{E}}(\mathcal{H}) < \mathcal{E}$;

(1*b*) $\mathcal{N}_{\mathcal{E}}(\mathcal{H}) \leq \mathcal{N}_{\mathcal{A}}(\mathcal{H}) < \mathcal{A}$;

(2) $\mathcal{T}(\mathcal{H}) \leq \mathcal{T}(\mathcal{G}) < \mathcal{T} < \mathcal{E}$;

(3) $\mathcal{T}(\mathcal{G}) \leq \mathcal{G} \leq \mathcal{N}_{\mathcal{E}}(\mathcal{G}) \leq \mathcal{N}_{\mathcal{A}}(\mathcal{G}) < \mathcal{A}$.

The subgroup $\mathcal{H}$ may be a *translationengleiche* or a *klassengleiche* or a general subgroup of $\mathcal{G}$. In any case, the normalizer $\mathcal{N}_{\mathcal{G}}(\mathcal{H})$ determines the length of the conjugacy class of $\mathcal{H} < \mathcal{G}$, but it is not feasible to list for each group–subgroup pair $\mathcal{H} < \mathcal{G}$ its normalizer $\mathcal{N}_{\mathcal{G}}(\mathcal{H})$. Indeed, it is only necessary to list for any space group $\mathcal{H}$ its normalizer $\mathcal{N}_{\mathcal{E}}(\mathcal{H})$ in the Euclidean group $\mathcal{E}$ of all isometries, as is done in *IT* A, Section 15.2.1. From such a list the normalizers for the group–subgroup pairs can be obtained easily, because for any chain of space groups $\mathcal{H} < \mathcal{G} < \mathcal{E}$, the relations $\mathcal{H} \leq \mathcal{N}_{\mathcal{G}}(\mathcal{H}) \leq \mathcal{G}$ and $\mathcal{H} \leq \mathcal{N}_{\mathcal{G}}(\mathcal{H}) \leq \mathcal{N}_{\mathcal{E}}(\mathcal{H})$ hold. The normalizer $\mathcal{N}_{\mathcal{G}}(\mathcal{H})$ consists consequently of all those isometries of $\mathcal{N}_{\mathcal{E}}(\mathcal{H})$ that are also elements of $\mathcal{G}$, *i.e.* that belong to the intersection $\mathcal{N}_{\mathcal{E}}(\mathcal{H}) \cap \mathcal{G}$, *cf.* the examples of Section 1.2.7.[10]

The isomorphism type of the Euclidean normalizer $\mathcal{N}_{\mathcal{E}}(\mathcal{H})$ may depend on the lattice parameters of the space group (*specialized* Euclidean normalizer). For example, if the lattice of the space group $P\overline{1}$ of a triclinic crystal is accidentally monoclinic at a certain temperature and pressure or for a certain composition in a continuous solid-solution series, then the Euclidean normalizer of this space group belongs to the space-group types $P2/m$ or $C2/m$, otherwise it belongs to $P\overline{1}$. Such a *specialized Euclidean normalizer* (here $P2/m$ or $C2/m$) may be distinguished from the *typical Euclidean normalizer* (here $P\overline{1}$), for which the lattice of $\mathcal{H}$ is not more symmetric than is required by the symmetry of $\mathcal{H}$. The specialized Euclidean normalizers were first listed in the 5th edition of *IT* A (2002), Section 15.2.1.

1.2.7. Application to domain structures

1.2.7.1. *Introductory remarks*

In this section, the group-theoretical aspects of domain (twin) formation (domain structure, transformation twin) from a homogeneous single crystal (phase **A**, parent phase) to a crystalline phase **B** (daughter phase, deformed phase) are discussed, where the space group $\mathcal{H}$ of phase **B** is a subgroup of the space group $\mathcal{G}$ of phase **A**, $\mathcal{H} < \mathcal{G}$. This happens, *e.g.*, in a displacive or order–disorder phase transition. In most cases phase **B**, the *domain structure*, is inhomogeneous, consisting of homogeneous regions which are called *domains*, defined below.

Only the basic group-theoretical relations are considered here. A deeper discussion of domain structures and their properties needs methods using representation theory, thermodynamic points of view (Landau theory), lattice dynamics and tensor properties of crystals. Such treatments are beyond the scope of this section. A detailed discussion of them is given by Tolédano *et al.* (2003) and by Janovec & Přívratská (2003).

In order to make the group-theoretical treatment possible, the *parent-clamping approximation*, abbreviated PCA, is introduced, by which the lattice parameters of phase **A** are not allowed to change at and after the transition to phase **B**, *cf.* Janovec & Přívratská (2003). Under the assumption of the PCA, two essential conditions hold:

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[10] For *maximal* subgroups, a calculation of the conjugacy classes is not necessary because these are indicated in the subgroup tables of Part 2 of this volume by braces to the left of the data sets for the low-index subgroups and by text for the series of isomorphic subgroups. For non-maximal subgroups, the conjugacy relations are not indicated but can be calculated in the way described here. They are also available online on the Bilbao crystallographic server, http://www.cryst.ehu.es/, under the program *Subgroupgraph*.

references