International Tables for Crystallography (2011). Vol. A1, Section 1.4.2.3, p. 28.

tuple of n vectors $\mathbf{B} := (\mathbf{a}_1, \ldots, \mathbf{a}_n)$ such that every vector of $\mathbf{V}$ can be written uniquely as a linear combination of the basis vectors: $\mathbf{V} = \{\mathbf{x} = x_1\mathbf{a}_1 + \ldots + x_n\mathbf{a}_n \mid x_1, \ldots, x_n \in \mathbb{R}\}$. Whereas a vector space has many different bases, the number n of vectors of a basis is uniquely determined and is called the *dimension* of $\mathbf{V}$. The isomorphism (see Section 1.4.3.4 for a definition of isomorphism) $\varphi_\mathbf{B}$ between $\mathbf{V}$ and $\mathbf{V}_n$ maps the vector $\mathbf{x} = x_1\mathbf{a}_1 + \ldots + x_n\mathbf{a}_n \in \mathbf{V}$ to its coefficient column

$$\boldsymbol{x} = \begin{pmatrix} x_1 \\ \vdots \\ x_n \end{pmatrix} \in \mathbf{V}_n$$

with respect to the chosen basis $\mathbf{B}$. The mapping $\varphi_\mathbf{B}$ respects addition of vectors and multiplication of vectors with real numbers. Moreover, $\varphi_\mathbf{B}$ is a bijective mapping, which means that for any coefficient column $\boldsymbol{x} \in \mathbf{V}_n$ there is a unique vector $\mathbf{x} \in \mathbf{V}$ with $\varphi_\mathbf{B}(\mathbf{x}) = \boldsymbol{x}$. Therefore one can perform all calculations using the coefficient columns.

An important concept in mathematics is the *automorphism group* of an object. In general, if one has an object (here the vector space $\mathbf{V}$) together with a structure (here the addition of vectors and the multiplication of vectors with real numbers), its automorphism group is the set of all one-to-one mappings of the object onto itself that preserve the structure.

A bijective mapping $\varphi : \mathbf{V} \to \mathbf{V}$ of the vector space $\mathbf{V}$ into itself satisfying $\varphi(\mathbf{v} + \mathbf{w}) = \varphi(\mathbf{v}) + \varphi(\mathbf{w})$ for all $\mathbf{v}, \mathbf{w} \in \mathbf{V}$ and $\varphi(x\mathbf{v}) = x\varphi(\mathbf{v})$ for all real numbers $x \in \mathbb{R}$ and all vectors $\mathbf{v} \in \mathbf{V}$ is called a *linear mapping* and the set of all these linear mappings is the *linear group* of $\mathbf{V}$. To know the image of $\mathbf{x} = x_1\mathbf{a}_1 + \ldots + x_n\mathbf{a}_n$ under a linear mapping φ it suffices to know the images of the basis vectors $\mathbf{a}_1, \ldots, \mathbf{a}_n$ under φ, since $\varphi(\mathbf{x}) = x_1\varphi(\mathbf{a}_1) + \ldots + x_n\varphi(\mathbf{a}_n)$. Writing the coefficient columns of the images of the basis vectors as columns of a matrix $\boldsymbol{A}$ [*i.e.* $\varphi(\mathbf{a}_i) = \sum_{j=1}^n \mathbf{a}_j A_{ji}$, $i = 1, \ldots, n$], then the coefficient column of $\varphi(\mathbf{x})$ with respect to the chosen basis $\mathbf{B}$ is just $\boldsymbol{Ax}$. Note that the matrix of a linear mapping depends on the basis $\mathbf{B}$ of $\mathbf{V}$. The matrix that corresponds to the composition of two linear mappings is the product of the two corresponding matrices. We have thus seen that the linear group of a vector space $\mathbf{V}$ of dimension n is isomorphic to the group of all invertible $(n \times n)$ matrices *via* the isomorphism $\varphi_\mathbf{B}$ that associates to a linear mapping its corresponding matrix (with respect to the basis $\mathbf{B}$). This means that one can perform all calculations with linear mappings using matrix calculations.

In crystallography, the translation-vector space has an additional structure: one can measure lengths and angles between vectors. An n-dimensional real vector space with such an additional structure is called a *Euclidean vector space*, $\mathbf{E}_n$. Its automorphism group is the set of all (bijective) linear mappings of $\mathbf{E}_n$ onto itself that preserve lengths and angles and is called the *orthogonal group* $\mathcal{O}_n$ of $\mathbf{E}_n$. If one chooses the basis $\mathbf{B} = (\mathbf{e}_1, \ldots, \mathbf{e}_n)$ to be the unit vectors (which are orthogonal vectors of length 1), then the isomorphism $\varphi_\mathbf{B}$ above maps the orthogonal group $\mathcal{O}_n$ onto the set of all $(n \times n)$ matrices $\boldsymbol{A}$ with $\boldsymbol{A}^\mathrm{T}\boldsymbol{A} = \boldsymbol{I}$, the $(n \times n)$ unit matrix. $^\mathrm{T}$ denotes the transposition operator, which maps columns to rows and rows to columns.

1.4.2.3. *The affine space*

In this section we build up a model for the 'point space'. Let us first assume $n = 2$. Then the affine space $\mathbb{A}_2$ may be imagined as an infinite sheet of paper parallel, let us say, to the $(\mathbf{a}, \mathbf{b})$ plane and cutting the $\mathbf{c}$ axis at $x_3 = 1$ in crystallographic notation. The points of $\mathbb{A}_2$ have coordinates

$$\begin{pmatrix} x_1 \\ x_2 \\ \hline 1 \end{pmatrix},$$

which are the coefficients of the vector from the origin to the point.

This observation is generalized by the following:

Definition 1.4.2.3.1.

$$\mathbb{A}_n := \{\begin{pmatrix} x_1 \\ \vdots \\ x_n \\ \hline 1 \end{pmatrix} \mid x_i \in \mathbb{R}\}$$

is an n-dimensional *affine space*. □

If

$$P = \begin{pmatrix} x_1 \\ \vdots \\ x_n \\ \hline 1 \end{pmatrix} \text{ and } Q = \begin{pmatrix} y_1 \\ \vdots \\ y_n \\ \hline 1 \end{pmatrix} \in \mathbb{A}_n,$$

then the vector $\overrightarrow{PQ}$ is defined as the difference

$$Q - P = \begin{pmatrix} y_1 - x_1 \\ \vdots \\ y_n - x_n \\ \hline 0 \end{pmatrix}$$

(computed in the vector space $\mathbf{V}_{n+1}$). The set of all $\overrightarrow{PQ}$ with $P, Q \in \mathbb{A}_n$ forms an n-dimensional vector space which is called the *underlying vector space* $\tau(\mathbb{A}_n)$. Omitting the last coefficient, we can identify

$$\tau(\mathbb{A}_n) = \{\begin{pmatrix} x_1 \\ \vdots \\ x_n \\ \hline 0 \end{pmatrix} \mid x_1, \ldots, x_n \in \mathbb{R}\}$$

with $\mathbf{V}_n$. As the coordinates already indicate, the sets $\mathbb{A}_n$ as well as $\tau(\mathbb{A}_n)$ can be viewed as subsets of $\mathbf{V}_{n+1}$. Computed in $\mathbf{V}_{n+1}$, the sum of two elements in $\tau(\mathbb{A}_n)$ is again in $\tau(\mathbb{A}_n)$, since the last coefficient of the sum is $0 + 0 = 0$ and the sum of a point $P \in \mathbb{A}_n$ and a vector $\mathbf{v} \in \mathbf{V}_n$ is again a point in $\mathbb{A}_n$ (since the last coordinate is $1 + 0 = 1$), but the sum of two points does not make sense.

1.4.2.4. *The affine group*

The affine group of geometry is the set of all mappings of the point space which fulfil the conditions

(1) parallel straight lines are mapped onto parallel straight lines;

(2) collinear points are mapped onto collinear points and the ratio of distances between them remains constant.