Theorem 1.4.3.3.2. (Sylow) If $|\mathcal{G}| = p^\alpha s$ for some prime p not dividing s, then all subgroups of order p^α of $\mathcal{G}$ are conjugate in $\mathcal{G}$. Such a subgroup $\mathcal{U} \leq \mathcal{G}$ of order $|\mathcal{U}| = p^\alpha$ is called a *Sylow p-subgroup*. □

Combining these two theorems with Theorem 1.4.3.2.8, one gets Sylow's third theorem:

Theorem 1.4.3.3.3. (Sylow) The number of Sylow p-subgroups of $\mathcal{G}$ is $\equiv 1 \pmod p$ and divides the order of $\mathcal{G}$. □

Proofs of the three theorems above can be found in Ledermann (1976), pp. 158–164, or in Ledermann & Weir (1996), pp. 155–161.

1.4.3.4. Isomorphisms

If one wants to compare objects such as groups or $\mathcal{G}$-sets, to be able to say when they should be considered equal, the concept of isomorphisms should be used:

Definition 1.4.3.4.1. Let $\mathcal{G}$ and $\mathcal{H}$ be groups and M and N be $\mathcal{G}$-sets.

(*a*) A *homomorphism* $\varphi : \mathcal{G} \to \mathcal{H}$ is a mapping of the set $\mathcal{G}$ into the set $\mathcal{H}$ respecting the composition law *i.e.* $\varphi(gh) = \varphi(g)\varphi(h)$ for all $g, h \in \mathcal{G}$.

If φ is bijective, it is called an *isomorphism* and one says $\mathcal{G}$ is *isomorphic* to $\mathcal{H}$ ($\mathcal{G} \cong \mathcal{H}$).

If $e \in \mathcal{H}$ is the unit element of $\mathcal{H}$, then the set of all pre-images of e is called the *kernel* of φ: $\ker(\varphi) := \{g \in \mathcal{G} \mid \varphi(g) = e\}$. An isomorphism $\varphi : \mathcal{G} \to \mathcal{G}$ is called an *automorphism* of $\mathcal{G}$.

(*b*) M and N are called *isomorphic* $\mathcal{G}$-sets if there is a bijection $\varphi : M \to N$ with $g \cdot \varphi(m) = \varphi(g \cdot m)$ for all $g \in \mathcal{G}, m \in M$. □

Example 1.4.3.4.2. In Example 1.4.3.1.3, the group homomorphism $\mathbb{Z} \to p\mathbb{Z}$ defined by $1 \mapsto p$ is a group isomorphism (from the group $\mathbb{Z}$ onto its subgroup $p\mathbb{Z}$).

Example 1.4.3.4.3. For any group element $g \in \mathcal{G}$, conjugation by g defines an automorphism of $\mathcal{G}$. In particular, if $\mathcal{U}$ is a subgroup of $\mathcal{G}$, then $\mathcal{U}$ and its conjugate subgroup $g\mathcal{U}g^{-1}$ are isomorphic.

Philosophy: If $\mathcal{G}$ and $\mathcal{H}$ are isomorphic groups, then all group-theoretical properties of $\mathcal{G}$ and $\mathcal{H}$ are the same. The calculations in $\mathcal{G}$ can be translated by the isomorphism to calculations in $\mathcal{H}$. Sometimes it is easier to calculate in one group than in the other and translate the result back *via* the inverse of the isomorphism. For example, the isomorphism between $\tau(\mathbb{A}_n)$ and $\mathbf{V}_n$ in Section 1.4.2 is an isomorphism of groups. It even respects scalar multiplication with real numbers, so in fact it is an isomorphism of vector spaces. While the composition of translations is more concrete and easier to imagine, the calculation of the resulting vector is much easier in $\mathbf{V}_n$. The concept of isomorphism says that you can translate to the more convenient group for your calculations and translate back afterwards without losing anything.

Note that a homomorphism is injective, hence an isomorphism onto its image, if and only if its kernel is trivial ($= \{e\}$).

Example 1.4.3.4.4

The mapping μ from the space $\tau(\mathbb{A}_n)$ of translation vectors into the affine group $\mathcal{A}_n$ defined by

$$\mu(\boldsymbol{w}) = \left(\begin{array}{c|c} \boldsymbol{I} & \boldsymbol{w} \\ \hline \boldsymbol{o}^{\mathrm{T}} & 1 \end{array}\right)$$

is a homomorphism of the group $\tau(\mathbb{A}_n)$ into $\mathcal{A}_n$. The kernel of this homomorphism is $\{\boldsymbol{o}\}$ and the image of the mapping is the translation subgroup $\mathcal{T}_n$ of $\mathcal{A}_n$. Hence the groups $\tau(\mathbb{A}_n)$ and $\mathcal{T}_n$ are isomorphic.

The affine group acts (as group of group automorphisms) on the normal subgroup $\mathcal{T}_n \trianglelefteq \mathcal{A}_n$ *via* conjugation: $g \cdot t := gtg^{-1}$. We have seen already in Example 1.4.3.2.4 (*b*) that it also acts (as a group of linear mappings) on $\tau(\mathbb{A}_n)$. The mapping μ is an isomorphism of $\mathcal{A}_n$-sets.

1.4.3.5. Isomorphism theorems

[*cf.* Ledermann (1976), pp. 68–73, or Ledermann & Weir (1996), pp. 85–92.]

Remark. If φ is a homomorphism $\mathcal{G} \to \mathcal{H}$ and $\mathcal{N} \trianglelefteq \mathcal{H}$ is a normal subgroup of $\mathcal{H}$, then the pre-image $\varphi^{-1}(\mathcal{N}) := \{g \in \mathcal{G} \mid \varphi(g) \in \mathcal{N}\}$ is a normal subgroup of $\mathcal{G}$. In particular, it holds that $\ker(\varphi) \trianglelefteq \mathcal{G}$.

Hence the factor group $\mathcal{G}/\ker(\varphi)$ is a well defined group. The following theorem says that this group is isomorphic to the image $\varphi(\mathcal{G}) \leq \mathcal{H}$ of φ:

Theorem 1.4.3.5.1. (First isomorphism theorem.) Let $\varphi : \mathcal{G} \to \mathcal{H}$ be a homomorphism of groups. Then

$$\overline{\varphi} : \mathcal{G}/\ker(\varphi) \to \varphi(\mathcal{G}) \leq \mathcal{H}$$

defined by $\overline{\varphi}(g\ker(\varphi)) = \varphi(g)$ is an isomorphism between the factor group $\mathcal{G}/\ker(\varphi)$ and the image group of φ, which is a subgroup of $\mathcal{H}$. □

For instance, if $\mathcal{R}$ is a space group and φ is mapping any element

$$\mathbb{W} = \left(\begin{array}{c|c} \boldsymbol{W} & \boldsymbol{w} \\ \hline \boldsymbol{o}^{\mathrm{T}} & 1 \end{array}\right) \in \mathcal{R}$$

to its linear part $\boldsymbol{W}$, then the kernel of φ is the translation group $\mathcal{T}(\mathcal{R})$ of $\mathcal{R}$ and the image is the point group $\overline{\mathcal{R}}$ of $\mathcal{R}$. The theorem says that the point group $\overline{\mathcal{R}}$ is isomorphic to the factor group $\mathcal{R}/\mathcal{T}(\mathcal{R})$.

Theorem 1.4.3.5.2. (Third isomorphism theorem.) Let $\mathcal{N} \trianglelefteq \mathcal{G}$ be a normal subgroup of the group $\mathcal{G}$ and $\mathcal{U} \leq \mathcal{G}$ be an arbitrary subgroup of $\mathcal{G}$. Then $\mathcal{U} \cap \mathcal{N} \trianglelefteq \mathcal{U}$ is a normal subgroup of $\mathcal{U}$ and

$$\mathcal{U}/(\mathcal{U} \cap \mathcal{N}) \cong \mathcal{N}\mathcal{U}/\mathcal{N}.$$

(For the definition of the group $\mathcal{N}\mathcal{U}$ see Proposition 1.4.3.2.11.) □

Definition 1.4.3.5.3. A subgroup $\mathcal{U} \leq \mathcal{H}$ is a *characteristic subgroup* $\mathcal{U}\,\mathrm{char}\,\mathcal{H}$ if $\varphi(\mathcal{U}) = \mathcal{U}$ for all automorphisms φ of $\mathcal{H}$. □

Remarks

(*a*) If $\mathcal{H}$ is a finite Abelian group and $\mathcal{P}$ is a Sylow p-subgroup of $\mathcal{H}$, then $\mathcal{P}\,\mathrm{char}\,\mathcal{H}$, because $\mathcal{P}$ is the only subgroup of $\mathcal{H}$ of order $|\mathcal{P}|$.

(*b*) If $\mathcal{H}$ is any group and $\mathcal{U}\,\mathrm{char}\,\mathcal{H}$, then $\mathcal{U} \trianglelefteq \mathcal{H}$ is also a normal subgroup of $\mathcal{H}$: for $h \in \mathcal{H}$ define the mapping $\kappa_h : \mathcal{H} \to \mathcal{H}$, $x \mapsto hxh^{-1}$. Then κ_h is an automorphism of $\mathcal{H}$ and $\kappa_h(\mathcal{U}) = h\mathcal{U}h^{-1} = \mathcal{U}$ since $\mathcal{U}$ is characteristic in $\mathcal{H}$.

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references