

1. SPACE GROUPS AND THEIR SUBGROUPS

One obtains the following:

Theorem 1.4.5.1.3. [Theorem of Galois (ca 1830).] Let \mathcal{H} be a finite group and let X be a faithful, primitive \mathcal{H} -set. Assume that $\{e\} \neq \mathcal{N} \trianglelefteq \mathcal{H}$ is an Abelian normal subgroup. Then

- (a) \mathcal{N} is a minimal normal subgroup of \mathcal{H} (i.e. for all $\mathcal{N}_1 \trianglelefteq \mathcal{H}$, $\mathcal{N}_1 \subseteq \mathcal{N} \Leftrightarrow \mathcal{N}_1 = \mathcal{N}$ or $\mathcal{N}_1 = \{e\}$).
- (b) \mathcal{N} is an elementary Abelian p -group for some prime p and $|X| = |\mathcal{N}|$ is a prime power.
- (c) $\mathcal{C}_{\mathcal{H}}(\mathcal{N}) = \mathcal{N}$ and \mathcal{N} is the unique minimal normal subgroup of \mathcal{H} . \square

Proof. Let $\{e\} \neq \mathcal{N} \trianglelefteq \mathcal{H}$ be an Abelian normal subgroup. Then \mathcal{N} acts faithfully and transitively on X . To establish a bijection between the sets \mathcal{N} and X , choose $x \in X$ and define $\varphi : \mathcal{N} \rightarrow X; n \mapsto n \cdot x$. Since \mathcal{N} is transitive, φ is surjective. To show the injectivity of φ , let $n_1, n_2 \in \mathcal{N}$ with $\varphi(n_1) = \varphi(n_2)$. Then $n_1 \cdot x = n_2 \cdot x$, hence $n_1^{-1}n_2x = x$. But then $n_1^{-1}n_2$ acts trivially on X , because if $y \in X$ then the transitivity of \mathcal{N} implies that there is an $n \in \mathcal{N}$ with $n \cdot x = y$. Then $n_1^{-1}n_2 \cdot y = n_1^{-1}n_2n \cdot x = nn_1^{-1}n_2 \cdot x = n \cdot x = y$, since \mathcal{N} is Abelian. Since X is a faithful \mathcal{H} -set, this implies $n_1^{-1}n_2 = e$ and therefore $n_1 = n_2$. This proves $|\mathcal{N}| = |X|$. Since this equality holds for all nontrivial Abelian normal subgroups of \mathcal{H} , statement (a) follows. If p is some prime dividing $|\mathcal{N}|$, then the Sylow p -subgroup of \mathcal{N} is normal in \mathcal{N} , since \mathcal{N} is Abelian. Therefore, it is also a characteristic subgroup of \mathcal{N} and hence a normal subgroup in \mathcal{H} (see the remarks below Definition 1.4.3.5.3). Since \mathcal{N} is a minimal normal subgroup of \mathcal{H} , this implies that \mathcal{N} is equal to its Sylow p -subgroup. Therefore, the order of \mathcal{N} is a prime power $|\mathcal{N}| = p^r$ for some prime p and $r \in \mathbb{N}$. Similarly, the set $\mathcal{N}^{p^r} := \{n^{p^r} \mid n \in \mathcal{N}\}$ is a normal subgroup of \mathcal{H} properly contained in \mathcal{N} . Therefore, $\mathcal{N}^{p^r} = \{e\}$ and \mathcal{N} is elementary Abelian. This establishes (b).

To see that (c) holds, let $g \in \mathcal{C}_{\mathcal{H}}(\mathcal{N})$. Choose $x \in X$. Then $g \cdot x = y \in X$. Since \mathcal{N} acts transitively, there is an $n \in \mathcal{N}$ such that $n \cdot x = y$. Hence $n^{-1}g \cdot x = x$. As above, let $z \in X$ be any element of X . Then there is an element $n_1 \in \mathcal{N}$ with $z = n_1 \cdot x$. Hence $n^{-1}g \cdot z = n^{-1}gn_1 \cdot x = n_1n^{-1}g \cdot x = n_1 \cdot x = z$. Since z was arbitrary and X is faithful, this implies that $g = n \in \mathcal{N}$. Therefore, $\mathcal{C}_{\mathcal{H}}(\mathcal{N}) \subseteq \mathcal{N}$. Since \mathcal{N} is Abelian, one has $\mathcal{N} \subseteq \mathcal{C}_{\mathcal{H}}(\mathcal{N})$, hence $\mathcal{N} = \mathcal{C}_{\mathcal{H}}(\mathcal{N})$. To see that \mathcal{N} is unique, let $\mathcal{P} \neq \mathcal{N}$ be another normal subgroup of \mathcal{H} . Since \mathcal{N} is a minimal normal subgroup, one has $\mathcal{N} \cap \mathcal{P} = \{e\}$, and, therefore, for $p \in \mathcal{P}$, $n \in \mathcal{N}$: $n^{-1}p^{-1}np \in \mathcal{N} \cap \mathcal{P} = \{e\}$. Hence \mathcal{P} centralizes \mathcal{N} , $\mathcal{P} \subseteq \mathcal{C}_{\mathcal{H}}(\mathcal{N}) = \mathcal{N}$, which is a contradiction. \square

Hence the groups \mathcal{H} that satisfy the hypotheses of the theorem of Galois are certain subgroups of an affine group $\mathcal{A}_n(\mathbb{Z}/p\mathbb{Z})$ over a finite field $\mathbb{Z}/p\mathbb{Z}$. This affine group is defined in a way similar to the affine group \mathcal{A}_n over the real numbers where one has to replace the real numbers by this finite field. Then \mathcal{N} is the translation subgroup of $\mathcal{A}_n(\mathbb{Z}/p\mathbb{Z})$ isomorphic to the n -dimensional vector space

$$(\mathbb{Z}/p\mathbb{Z})^n = \{x = \begin{pmatrix} x_1 \\ \vdots \\ x_n \end{pmatrix} \mid x_1, \dots, x_n \in \mathbb{Z}/p\mathbb{Z}\}$$

over $\mathbb{Z}/p\mathbb{Z}$. The set X is the corresponding affine space $\mathbb{A}_n(\mathbb{Z}/p\mathbb{Z})$. The factor group $\overline{\mathcal{H}} = \mathcal{H}/\mathcal{N}$ is isomorphic to a subgroup of the linear group of $(\mathbb{Z}/p\mathbb{Z})^n$ that does not leave invariant any nontrivial subspace of $(\mathbb{Z}/p\mathbb{Z})^n$.

1.4.5.2. Soluble groups

Definition 1.4.5.2.1. Let \mathcal{G} be a group. The *derived series* of \mathcal{G} is the series $(\mathcal{G}_0, \mathcal{G}_1, \dots)$ defined via $\mathcal{G}_0 := \mathcal{G}$, $\mathcal{G}_i := \langle g^{-1}h^{-1}gh \mid g, h \in \mathcal{G}_{i-1} \rangle$. The group \mathcal{G}_1 is called the *derived subgroup* of \mathcal{G} . The group \mathcal{G} is called *soluble* if $\mathcal{G}_n = \{e\}$ for some $n \in \mathbb{N}$. \square

Remarks

- (i) The \mathcal{G}_i are characteristic subgroups of \mathcal{G} .
- (ii) \mathcal{G} is Abelian if and only if $\mathcal{G}_1 = \{e\}$.
- (iii) \mathcal{G}_1 is characterized as the smallest normal subgroup of \mathcal{G} , such that $\mathcal{G}/\mathcal{G}_1$ is Abelian, in the sense that every normal subgroup of \mathcal{G} with an Abelian factor group contains \mathcal{G}_1 .
- (iv) Subgroups and factor groups of soluble groups are soluble.
- (v) If $\mathcal{N} \trianglelefteq \mathcal{G}$ is a normal subgroup, then \mathcal{G} is soluble if and only if \mathcal{G}/\mathcal{N} and \mathcal{N} are both soluble.

Example 1.4.5.2.2

The derived series of $Cyc_2 \times Sym_4$ is

$$Cyc_2 \times Sym_4 \supseteq Alt_4 \supseteq Cyc_2 \times Cyc_2 \supseteq \mathcal{I}$$

(or in Hermann–Mauguin notation $m\bar{3}m \supseteq 23 \supseteq 222 \supseteq 1$) and that of $Cyc_2 \times Cyc_2 \times Sym_3$ is

$$Cyc_2 \times Cyc_2 \times Sym_3 \supseteq Cyc_3 \supseteq \mathcal{I}$$

(Hermann–Mauguin notation: $6/mmm \supseteq 3 \supseteq 1$).

Hence these two groups are soluble. (For an explanation of the groups that occur here and later, see Section 1.4.3.6.)

Now let $\mathcal{R} \leq \mathcal{E}_3$ be a three-dimensional space group. Then $\mathcal{T}(\mathcal{R})$ is an Abelian normal subgroup, hence $\mathcal{T}(\mathcal{R})$ is soluble. The factor group $\mathcal{R}/\mathcal{T}(\mathcal{R})$ is isomorphic to a subgroup of either $Cyc_2 \times Sym_4$ or $Cyc_2 \times Cyc_2 \times Sym_3$ and therefore also soluble. Using the remark above, one deduces that all three-dimensional space groups are soluble.

Lemma 1.4.5.2.3. Let \mathcal{R} be a three-dimensional space group. Then \mathcal{R} is soluble. \square

1.4.5.3. Maximal subgroups of soluble groups

Now let \mathcal{G} be a soluble group and $\mathcal{M} \leq \mathcal{G}$ a maximal subgroup of finite index in \mathcal{G} . Then the set of left cosets $X := \mathcal{G}/\mathcal{M}$ is a primitive finite \mathcal{G} -set. Let $\mathcal{K} = \text{core}(\mathcal{M})$ be the kernel of the action of \mathcal{G} on X . Then the factor group $\mathcal{H} := \mathcal{G}/\mathcal{K}$ acts faithfully on X . In particular, \mathcal{H} is a finite group and X is a primitive, faithful \mathcal{H} -set. Since \mathcal{G} is soluble, the factor group \mathcal{H} is also a soluble group. Let $\mathcal{H} \supseteq \mathcal{H}_1 \supseteq \dots \supseteq \mathcal{H}_{n-1} \supseteq \{e\}$ be the derived series of \mathcal{H} with $\mathcal{N} := \mathcal{H}_{n-1} \neq \{e\}$. Then \mathcal{N} is an Abelian normal subgroup of \mathcal{H} . The theorem of Galois (Theorem 1.4.5.1.3) states that \mathcal{N} is an elementary Abelian p -group for some prime p and $|X| = |\mathcal{N}| = p^r$ for some $r \in \mathbb{N}$. Since $X = \mathcal{G}/\mathcal{M}$, the order of X is the index $[\mathcal{G} : \mathcal{M}]$ of \mathcal{M} in \mathcal{G} . Therefore one gets the following theorem:

Theorem 1.4.5.3.1. If $\mathcal{M} \leq \mathcal{G}$ is a maximal subgroup of finite index in the soluble group \mathcal{G} , then its index $[\mathcal{G} : \mathcal{M}]$ is a prime power. \square

In the proof of Theorem 1.4.5.1.3, we have established a bijection between \mathcal{N} and the \mathcal{H} -set X , which is now $X := \mathcal{G}/\mathcal{M}$. Taking the full pre-image