International Tables for Crystallography (2011). Vol. A1, Section 1.4.5.3, pp. 36–37.

One obtains the following:

Theorem 1.4.5.1.3. [Theorem of Galois (*ca* 1830).] Let $\mathcal{H}$ be a finite group and let X be a faithful, primitive $\mathcal{H}$-set. Assume that $\{e\} \neq \mathcal{N} \trianglelefteq \mathcal{H}$ is an Abelian normal subgroup. Then

(*a*) $\mathcal{N}$ is a minimal normal subgroup of $\mathcal{H}$ (*i.e.* for all $\mathcal{N}_1 \trianglelefteq \mathcal{H}$, $\mathcal{N}_1 \subseteq \mathcal{N} \Leftrightarrow \mathcal{N}_1 = \mathcal{N}$ or $\mathcal{N}_1 = \{e\}$).

(*b*) $\mathcal{N}$ is an elementary Abelian p-group for some prime p and $|X| = |\mathcal{N}|$ is a prime power.

(*c*) $C_{\mathcal{H}}(\mathcal{N}) = \mathcal{N}$ and $\mathcal{N}$ is the unique minimal normal subgroup of $\mathcal{H}$. □

Proof. Let $\{e\} \neq \mathcal{N} \trianglelefteq \mathcal{H}$ be an Abelian normal subgroup. Then $\mathcal{N}$ acts faithfully and transitively on X. To establish a bijection between the sets $\mathcal{N}$ and X, choose $x \in X$ and define $\varphi : \mathcal{N} \to X; n \mapsto n \cdot x$. Since $\mathcal{N}$ is transitive, φ is surjective. To show the injectivity of φ, let $n_1, n_2 \in \mathcal{N}$ with $\varphi(n_1) = \varphi(n_2)$. Then $n_1 \cdot x = n_2 \cdot x$, hence $n_1^{-1} n_2 x = x$. But then $n_1^{-1} n_2$ acts trivially on X, because if $y \in X$ then the transitivity of $\mathcal{N}$ implies that there is an $n \in \mathcal{N}$ with $n \cdot x = y$. Then $n_1^{-1} n_2 \cdot y = n_1^{-1} n_2 n \cdot x = n n_1^{-1} n_2 \cdot x = n \cdot x = y$, since $\mathcal{N}$ is Abelian. Since X is a faithful $\mathcal{H}$-set, this implies $n_1^{-1} n_2 = e$ and therefore $n_1 = n_2$. This proves $|\mathcal{N}| = |X|$. Since this equality holds for all nontrivial Abelian normal subgroups of $\mathcal{H}$, statement (*a*) follows. If p is some prime dividing $|\mathcal{N}|$, then the Sylow p-subgroup of $\mathcal{N}$ is normal in $\mathcal{N}$, since $\mathcal{N}$ is Abelian. Therefore, it is also a characteristic subgroup of $\mathcal{N}$ and hence a normal subgroup in $\mathcal{H}$ (see the remarks below Definition 1.4.3.5.3). Since $\mathcal{N}$ is a minimal normal subgroup of $\mathcal{H}$, this implies that $\mathcal{N}$ is equal to its Sylow p-subgroup. Therefore, the order of $\mathcal{N}$ is a prime power $|\mathcal{N}| = p^r$ for some prime p and $r \in \mathbb{N}$. Similarly, the set $\mathcal{N}^p := \{n^p \mid n \in \mathcal{N}\}$ is a normal subgroup of $\mathcal{H}$ properly contained in $\mathcal{N}$. Therefore, $\mathcal{N}^p = \{e\}$ and $\mathcal{N}$ is elementary Abelian. This establishes (*b*).

To see that (*c*) holds, let $g \in C_{\mathcal{H}}(\mathcal{N})$. Choose $x \in X$. Then $g \cdot x = y \in X$. Since $\mathcal{N}$ acts transitively, there is an $n \in \mathcal{N}$ such that $n \cdot x = y$. Hence $n^{-1} g \cdot x = x$. As above, let $z \in X$ be any element of X. Then there is an element $n_1 \in \mathcal{N}$ with $z = n_1 \cdot x$. Hence $n^{-1} g \cdot z = n^{-1} g n_1 \cdot x = n_1 n^{-1} g \cdot x = n_1 \cdot x = z$. Since z was arbitrary and X is faithful, this implies that $g = n \in \mathcal{N}$. Therefore, $C_{\mathcal{H}}(\mathcal{N}) \subseteq \mathcal{N}$. Since $\mathcal{N}$ is Abelian, one has $\mathcal{N} \subseteq C_{\mathcal{H}}(\mathcal{N})$, hence $\mathcal{N} = C_{\mathcal{H}}(\mathcal{N})$. To see that $\mathcal{N}$ is unique, let $\mathcal{P} \neq \mathcal{N}$ be another normal subgroup of $\mathcal{H}$. Since $\mathcal{N}$ is a minimal normal subgroup, one has $\mathcal{N} \cap \mathcal{P} = \{e\}$, and, therefore, for $p \in \mathcal{P}, n \in \mathcal{N}$: $n^{-1} p^{-1} n p \in \mathcal{N} \cap \mathcal{P} = \{e\}$. Hence $\mathcal{P}$ centralizes $\mathcal{N}$, $\mathcal{P} \subseteq C_{\mathcal{H}}(\mathcal{N}) = \mathcal{N}$, which is a contradiction. QED

Hence the groups $\mathcal{H}$ that satisfy the hypotheses of the theorem of Galois are certain subgroups of an affine group $\mathcal{A}_n(\mathbb{Z}/p\mathbb{Z})$ over a finite field $\mathbb{Z}/p\mathbb{Z}$. This affine group is defined in a way similar to the affine group $\mathcal{A}_n$ over the real numbers where one has to replace the real numbers by this finite field. Then $\mathcal{N}$ is the translation subgroup of $\mathcal{A}_n(\mathbb{Z}/p\mathbb{Z})$ isomorphic to the n-dimensional vector space

$$(\mathbb{Z}/p\mathbb{Z})^n = \{\boldsymbol{x} = \begin{pmatrix} x_1 \\ \vdots \\ x_n \end{pmatrix} \mid x_1, \ldots, x_n \in \mathbb{Z}/p\mathbb{Z}\}$$

over $\mathbb{Z}/p\mathbb{Z}$. The set X is the corresponding affine space $\mathbb{A}_n(\mathbb{Z}/p\mathbb{Z})$. The factor group $\overline{\mathcal{H}} = \mathcal{H}/\mathcal{N}$ is isomorphic to a subgroup of the linear group of $(\mathbb{Z}/p\mathbb{Z})^n$ that does not leave invariant any nontrivial subspace of $(\mathbb{Z}/p\mathbb{Z})^n$.

1.4.5.2. Soluble groups

Definition 1.4.5.2.1. Let $\mathcal{G}$ be a group. The *derived series* of $\mathcal{G}$ is the series $(\mathcal{G}_0, \mathcal{G}_1, \ldots)$ defined *via* $\mathcal{G}_0 := \mathcal{G}$, $\mathcal{G}_i := \langle g^{-1} h^{-1} g h \mid g, h \in \mathcal{G}_{i-1} \rangle$. The group $\mathcal{G}_1$ is called the *derived subgroup* of $\mathcal{G}$. The group $\mathcal{G}$ is called *soluble* if $\mathcal{G}_n = \{e\}$ for some $n \in \mathbb{N}$. □

Remarks

(i) The $\mathcal{G}_i$ are characteristic subgroups of $\mathcal{G}$.
(ii) $\mathcal{G}$ is Abelian if and only if $\mathcal{G}_1 = \{e\}$.
(iii) $\mathcal{G}_1$ is characterized as the smallest normal subgroup of $\mathcal{G}$, such that $\mathcal{G}/\mathcal{G}_1$ is Abelian, in the sense that every normal subgroup of $\mathcal{G}$ with an Abelian factor group contains $\mathcal{G}_1$.
(iv) Subgroups and factor groups of soluble groups are soluble.
(v) If $\mathcal{N} \trianglelefteq \mathcal{G}$ is a normal subgroup, then $\mathcal{G}$ is soluble if and only if $\mathcal{G}/\mathcal{N}$ and $\mathcal{N}$ are both soluble.

Example 1.4.5.2.2

The derived series of $\mathcal{C}yc_2 \times \mathcal{S}ym_4$ is

$$\mathcal{C}yc_2 \times \mathcal{S}ym_4 \trianglerighteq \mathcal{A}lt_4 \trianglerighteq \mathcal{C}yc_2 \times \mathcal{C}yc_2 \trianglerighteq \mathcal{I}$$

(or in Hermann–Mauguin notation $m\bar{3}m \trianglerighteq 23 \trianglerighteq 222 \trianglerighteq 1$) and that of $\mathcal{C}yc_2 \times \mathcal{C}yc_2 \times \mathcal{S}ym_3$ is

$$\mathcal{C}yc_2 \times \mathcal{C}yc_2 \times \mathcal{S}ym_3 \trianglerighteq \mathcal{C}yc_3 \trianglerighteq \mathcal{I}$$

(Hermann–Mauguin notation: $6/mmm \trianglerighteq 3 \trianglerighteq 1$).

Hence these two groups are soluble. (For an explanation of the groups that occur here and later, see Section 1.4.3.6.)

Now let $\mathcal{R} \leq \mathcal{E}_3$ be a three-dimensional space group. Then $\mathcal{T}(\mathcal{R})$ is an Abelian normal subgroup, hence $\mathcal{T}(\mathcal{R})$ is soluble. The factor group $\mathcal{R}/\mathcal{T}(\mathcal{R})$ is isomorphic to a subgroup of either $\mathcal{C}yc_2 \times \mathcal{S}ym_4$ or $\mathcal{C}yc_2 \times \mathcal{C}yc_2 \times \mathcal{S}ym_3$ and therefore also soluble. Using the remark above, one deduces that all three-dimensional space groups are soluble.

Lemma 1.4.5.2.3. Let $\mathcal{R}$ be a three-dimensional space group. Then $\mathcal{R}$ is soluble. □

1.4.5.3. Maximal subgroups of soluble groups

Now let $\mathcal{G}$ be a soluble group and $\mathcal{M} \leq \mathcal{G}$ a maximal subgroup of finite index in $\mathcal{G}$. Then the set of left cosets $X := \mathcal{G}/\mathcal{M}$ is a primitive finite $\mathcal{G}$-set. Let $\mathcal{K} = \mathrm{core}(\mathcal{M})$ be the kernel of the action of $\mathcal{G}$ on X. Then the factor group $\mathcal{H} := \mathcal{G}/\mathcal{K}$ acts faithfully on X. In particular, $\mathcal{H}$ is a finite group and X is a primitive, faithful $\mathcal{H}$-set. Since $\mathcal{G}$ is soluble, the factor group $\mathcal{H}$ is also a soluble group. Let $\mathcal{H} \trianglerighteq \mathcal{H}_1 \trianglerighteq \ldots \trianglerighteq \mathcal{H}_{n-1} \trianglerighteq \{e\}$ be the derived series of $\mathcal{H}$ with $\mathcal{N} := \mathcal{H}_{n-1} \neq \{e\}$. Then $\mathcal{N}$ is an Abelian normal subgroup of $\mathcal{H}$. The theorem of Galois (Theorem 1.4.5.1.3) states that $\mathcal{N}$ is an elementary Abelian p-group for some prime p and $|X| = |\mathcal{N}| = p^r$ for some $r \in \mathbb{N}$. Since $X = \mathcal{G}/\mathcal{M}$, the order of X is the index $[\mathcal{G} : \mathcal{M}]$ of $\mathcal{M}$ in $\mathcal{G}$. Therefore one gets the following theorem:

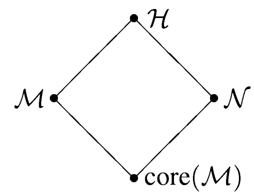
Theorem 1.4.5.3.1. If $\mathcal{M} \leq \mathcal{G}$ is a maximal subgroup of finite index in the soluble group $\mathcal{G}$, then its index $|\mathcal{G}/\mathcal{M}|$ is a prime power. □

In the proof of Theorem 1.4.5.1.3, we have established a bijection between $\mathcal{N}$ and the $\mathcal{H}$-set X, which is now $X := \mathcal{G}/\mathcal{M}$. Taking the full pre-image

$$\mathcal{N}' := \mathcal{N}\,\mathrm{core}\,(\mathcal{M})$$

of $\mathcal{N}$ in $\mathcal{G}$, then one has $\mathcal{G} = \mathcal{N}'\mathcal{M}$ and $\mathcal{M} \cap \mathcal{N}' = \mathrm{core}\,(\mathcal{M})$. Hence we have seen the first part of the following theorem:

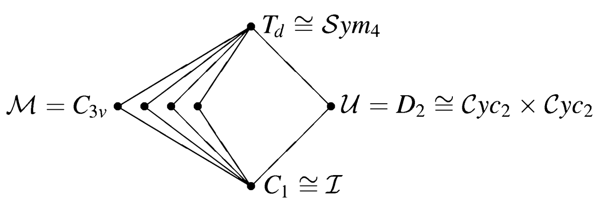
Theorem 1.4.5.3.2. Let $\mathcal{M} \leq \mathcal{G}$ be a maximal subgroup of the soluble group $\mathcal{G}$. Then the factor group $\mathcal{H} := \mathcal{G}/\mathrm{core}(\mathcal{M})$ acts primitively and faithfully on $X := \mathcal{G}/\mathcal{M}$, and there is a normal subgroup $\mathcal{N}' \trianglelefteq \mathcal{G}$ with $\mathcal{M}\mathcal{N}' = \mathcal{G}$ and $\mathcal{M} \cap \mathcal{N}' = \mathrm{core}(\mathcal{M})$. Moreover, if $\mathcal{M}'$ is another subgroup of $\mathcal{G}$, with $\mathcal{M}'\mathcal{N}' = \mathcal{G}$ and $\mathcal{M}' \cap \mathcal{N}' = \mathrm{core}(\mathcal{M})$, then $\mathcal{M}'$ is conjugate to $\mathcal{M}$.



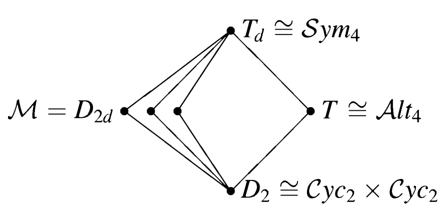
□

Example 1.4.5.3.3

$\mathcal{G} = \mathcal{S}ym_4 \cong T_d$ is the tetrahedral group from Section 1.4.3.2 and $\mathcal{S}ym_3 \cong \mathcal{M} = C_{3v} \leq \mathcal{G}$ is the stabilizer of one of the four apices in the tetrahedron. Then $\mathrm{core}(\mathcal{M}) = \{e\}$ and $\mathcal{G}/\mathcal{M}$ is a faithful $\mathcal{G}$-set which can be identified with the set of apices of the tetrahedron. The normal subgroup $\mathcal{N} = \mathcal{N}'$ is the normal subgroup $\mathcal{U}$ of Section 1.4.3.2.



Now let $\mathcal{G} = \mathcal{S}ym_4 \cong T_d$ be as above, and take $D_{2d} \cong \mathcal{M} \leq \mathcal{G}$ a Sylow 2-subgroup of $\mathcal{G}$. Then $\mathrm{core}(\mathcal{M}) = D_2 \cong \mathcal{C}yc_2 \times \mathcal{C}yc_2$ is the normal subgroup $\mathcal{U}$ from Section 1.4.3.2 and $\mathcal{H} = \mathcal{G}/\mathrm{core}(\mathcal{M}) \cong \mathcal{S}ym_3$.



These observations result in an algorithm for computing maximal subgroups of soluble groups $\mathcal{G}$:

(1) compute normal subgroups $\mathcal{C}$ [candidates for $\mathrm{core}(\mathcal{M})$];
(2) compute a minimal normal subgroup $\mathcal{N}/\mathcal{C}$ of $\mathcal{G}/\mathcal{C}$;
(3) find $\mathcal{M}/\mathcal{C}$ as a complement of $\mathcal{N}/\mathcal{C}$ in $\mathcal{G}/\mathcal{C}$.

1.4.6. Quantitative results

This section gives estimates for the number of maximal subgroups of a given index in space groups.

1.4.6.1. General results

The first very easy but useful remark applies to general groups $\mathcal{G}$:

Remark. Let $\mathcal{M} \leq \mathcal{G}$ be a maximal subgroup of $\mathcal{G}$ of finite index $i := [\mathcal{G} : \mathcal{M}] < \infty$. Then $\mathcal{M} \leq \mathcal{N}_{\mathcal{G}}(\mathcal{M}) \leq \mathcal{G}$. Hence the maximality of $\mathcal{M}$ implies that either $\mathcal{N}_{\mathcal{G}}(\mathcal{M}) = \mathcal{G}$ and $\mathcal{M}$ is a normal subgroup of $\mathcal{G}$ or $\mathcal{N}_{\mathcal{G}}(\mathcal{M}) = \mathcal{M}$ and $\mathcal{G}$ has i maximal subgroups that are conjugate to $\mathcal{M}$.

The smallest possible index of a proper subgroup is 2. It is well known and easy to see that subgroups of index 2 are normal subgroups:

Proposition 1.4.6.1.1. Let $\mathcal{G}$ be a group and $\mathcal{M} \leq \mathcal{G}$ a subgroup of index $2 = |\mathcal{G}/\mathcal{M}|$. Then $\mathcal{M}$ is a normal subgroup of $\mathcal{G}$. □

Proof. Choose an element $g \in \mathcal{G}$, $g \notin \mathcal{M}$. Then $\mathcal{G} = \mathcal{M} \cup g\mathcal{M} = \mathcal{M} \cup \mathcal{M}g$. Hence $g\mathcal{M} = \mathcal{M}g$ and therefore $g\mathcal{M}g^{-1} = \mathcal{M}$. Since this is also true if $g \in \mathcal{M}$, the proposition follows. QED

Let $\mathcal{M}$ be a subgroup of a group $\mathcal{G}$ of index 2. Then $\mathcal{M} \trianglelefteq \mathcal{G}$ is a normal subgroup and the factor group $\mathcal{G}/\mathcal{M}$ is a group of order 2. Since groups of order 2 are Abelian, it follows that the derived subgroup $\mathcal{G}_1$ of $\mathcal{G}$ (*cf*. Definition 1.4.5.2.1) (which is the smallest normal subgroup of $\mathcal{G}$ such that the factor group is Abelian) is contained in $\mathcal{M}$. Hence all maximal subgroups of index 2 in $\mathcal{G}$ contain $\mathcal{G}_1$. If one defines $\mathcal{N} := \cap\{\mathcal{M} \leq \mathcal{G} \mid [\mathcal{G} : \mathcal{M}] = 2\}$, then $\mathcal{G}/\mathcal{N}$ is an elementary Abelian 2-group and hence a vector space over the field with two elements. The maximal subgroups of $\mathcal{G}/\mathcal{N}$ are the maximal subspaces of this vector space, hence their number is $2^a - 1$, where $a := \dim_{\mathbb{Z}/2\mathbb{Z}}(\mathcal{G}/\mathcal{N})$.

This shows the following:

Corollary 1.4.6.1.2. The number of subgroups of $\mathcal{G}$ of index 2 is of the form $2^a - 1$ for some $a \geq 0$. □

Dealing with subgroups of index 3, one has the following:

Proposition 1.4.6.1.3. Let $\mathcal{U}$ be a subgroup of the group $\mathcal{G}$ with $[\mathcal{G} : \mathcal{U}] = 3$. Then $\mathcal{U}$ is either a normal subgroup of $\mathcal{G}$ or $\mathcal{G}/\mathrm{core}(\mathcal{U}) \cong \mathcal{S}_3$ and there are three subgroups of $\mathcal{G}$ conjugate to $\mathcal{U}$. □

Proof. $\mathcal{G}/\mathrm{core}\,(\mathcal{U})$ is isomorphic to a subgroup of $\mathcal{S}ym_3$ that acts primitively on $\{1, 2, 3\}$. Hence either $\mathcal{G}/\mathrm{core}\,(\mathcal{U}) \cong \mathcal{C}yc_3$ and $\mathcal{U} = \mathrm{core}\,(\mathcal{U})$ is a normal subgroup of $\mathcal{G}$ or $\mathcal{G}/\mathrm{core}\,(\mathcal{U}) \cong \mathcal{S}ym_3$, $\mathcal{U}/\mathrm{core}\,(\mathcal{U}) \cong \mathcal{C}yc_2$ and there are three subgroups of $\mathcal{G}$ conjugate to $\mathcal{U}$. QED

1.4.6.2. Three-dimensional space groups

We now come to space groups. By Lemma 1.4.5.2.3, all three-dimensional space groups are soluble. Theorem 1.4.5.3.1 says that the index of a maximal subgroup of a soluble group is a prime power (or infinite). Since the index of a maximal subgroup of a space group is always finite (see Corollary 1.4.4.2.7), we get:

Corollary 1.4.6.2.1. Let $\mathcal{G}$ be a three-dimensional space group and $\mathcal{M} \leq \mathcal{G}$ a maximal subgroup. Then $[\mathcal{G} : \mathcal{M}]$ is a prime power. □

Let $\mathcal{R}$ be a three-dimensional space group and $\mathcal{P} = \mathcal{R}/\mathcal{T}(\mathcal{R})$ its point group. It is well known that the order of $\mathcal{P}$ is of the form 2^a3^b with $a = 0, 1, 2, 3$ or 4 and $b = 0, 1$. By Corollary 1.4.4.2.4, the t-subgroups of $\mathcal{R}$ are in one-to-one correspondence with the subgroups of $\mathcal{P}$. Let us look at the t-subgroups of $\mathcal{R}$ of index 3. It is clear that $\mathcal{P}$ has no subgroup of index 3 if $b = 0$, since the index of a subgroup divides the order of the finite group $\mathcal{P}$ by the theorem of Lagrange. If $b = 1$, then any subgroup $\mathcal{S}$ of $\mathcal{P}$ of index 3 has order $|\mathcal{P}|/3 = 2^a$ and hence is a Sylow 2-subgroup of $\mathcal{P}$. Therefore there is such a subgroup $\mathcal{S}$ of index 3 in $\mathcal{P}$ by the first theorem of Sylow, Theorem 1.4.3.3.1. By the second theorem of Sylow, Theorem 1.4.3.3.2, all these Sylow 2-subgroups of $\mathcal{P}$ are