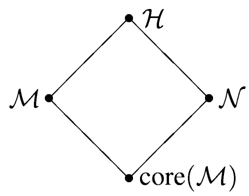
International Tables for Crystallography (2011). Vol. A1, Section 1.4.6.1, p. 37.

$$\mathcal{N}' := \mathcal{N}\,\mathrm{core}\,(\mathcal{M})$$

of $\mathcal{N}$ in $\mathcal{G}$, then one has $\mathcal{G} = \mathcal{N}'\mathcal{M}$ and $\mathcal{M} \cap \mathcal{N}' = \mathrm{core}\,(\mathcal{M})$. Hence we have seen the first part of the following theorem:

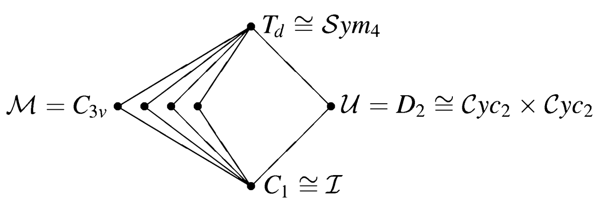
Theorem 1.4.5.3.2. Let $\mathcal{M} \leq \mathcal{G}$ be a maximal subgroup of the soluble group $\mathcal{G}$. Then the factor group $\mathcal{H} := \mathcal{G}/\mathrm{core}(\mathcal{M})$ acts primitively and faithfully on $X := \mathcal{G}/\mathcal{M}$, and there is a normal subgroup $\mathcal{N}' \trianglelefteq \mathcal{G}$ with $\mathcal{M}\mathcal{N}' = \mathcal{G}$ and $\mathcal{M} \cap \mathcal{N}' = \mathrm{core}(\mathcal{M})$. Moreover, if $\mathcal{M}'$ is another subgroup of $\mathcal{G}$, with $\mathcal{M}'\mathcal{N}' = \mathcal{G}$ and $\mathcal{M}' \cap \mathcal{N}' = \mathrm{core}(\mathcal{M})$, then $\mathcal{M}'$ is conjugate to $\mathcal{M}$.



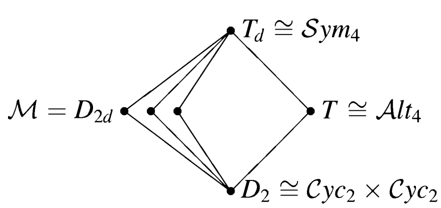
□

Example 1.4.5.3.3

$\mathcal{G} = \mathcal{S}ym_4 \cong T_d$ is the tetrahedral group from Section 1.4.3.2 and $\mathcal{S}ym_3 \cong \mathcal{M} = C_{3v} \leq \mathcal{G}$ is the stabilizer of one of the four apices in the tetrahedron. Then $\mathrm{core}(\mathcal{M}) = \{e\}$ and $\mathcal{G}/\mathcal{M}$ is a faithful $\mathcal{G}$-set which can be identified with the set of apices of the tetrahedron. The normal subgroup $\mathcal{N} = \mathcal{N}'$ is the normal subgroup $\mathcal{U}$ of Section 1.4.3.2.



Now let $\mathcal{G} = \mathcal{S}ym_4 \cong T_d$ be as above, and take $D_{2d} \cong \mathcal{M} \leq \mathcal{G}$ a Sylow 2-subgroup of $\mathcal{G}$. Then $\mathrm{core}(\mathcal{M}) = D_2 \cong \mathcal{C}yc_2 \times \mathcal{C}yc_2$ is the normal subgroup $\mathcal{U}$ from Section 1.4.3.2 and $\mathcal{H} = \mathcal{G}/\mathrm{core}(\mathcal{M}) \cong \mathcal{S}ym_3$.



These observations result in an algorithm for computing maximal subgroups of soluble groups $\mathcal{G}$:

(1) compute normal subgroups $\mathcal{C}$ [candidates for core($\mathcal{M}$)];
(2) compute a minimal normal subgroup $\mathcal{N}/\mathcal{C}$ of $\mathcal{G}/\mathcal{C}$;
(3) find $\mathcal{M}/\mathcal{C}$ as a complement of $\mathcal{N}/\mathcal{C}$ in $\mathcal{G}/\mathcal{C}$.

1.4.6. Quantitative results

This section gives estimates for the number of maximal subgroups of a given index in space groups.

1.4.6.1. General results

The first very easy but useful remark applies to general groups $\mathcal{G}$:

Remark. Let $\mathcal{M} \leq \mathcal{G}$ be a maximal subgroup of $\mathcal{G}$ of finite index $i := [\mathcal{G} : \mathcal{M}] < \infty$. Then $\mathcal{M} \leq \mathcal{N}_{\mathcal{G}}(\mathcal{M}) \leq \mathcal{G}$. Hence the maximality of $\mathcal{M}$ implies that either $\mathcal{N}_{\mathcal{G}}(\mathcal{M}) = \mathcal{G}$ and $\mathcal{M}$ is a normal subgroup of $\mathcal{G}$ or $\mathcal{N}_{\mathcal{G}}(\mathcal{M}) = \mathcal{M}$ and $\mathcal{G}$ has i maximal subgroups that are conjugate to $\mathcal{M}$.

The smallest possible index of a proper subgroup is 2. It is well known and easy to see that subgroups of index 2 are normal subgroups:

Proposition 1.4.6.1.1. Let $\mathcal{G}$ be a group and $\mathcal{M} \leq \mathcal{G}$ a subgroup of index $2 = |\mathcal{G}/\mathcal{M}|$. Then $\mathcal{M}$ is a normal subgroup of $\mathcal{G}$. □

Proof. Choose an element $g \in \mathcal{G}$, $g \notin \mathcal{M}$. Then $\mathcal{G} = \mathcal{M} \cup g\mathcal{M} = \mathcal{M} \cup \mathcal{M}g$. Hence $g\mathcal{M} = \mathcal{M}g$ and therefore $g\mathcal{M}g^{-1} = \mathcal{M}$. Since this is also true if $g \in \mathcal{M}$, the proposition follows. QED

Let $\mathcal{M}$ be a subgroup of a group $\mathcal{G}$ of index 2. Then $\mathcal{M} \trianglelefteq \mathcal{G}$ is a normal subgroup and the factor group $\mathcal{G}/\mathcal{M}$ is a group of order 2. Since groups of order 2 are Abelian, it follows that the derived subgroup $\mathcal{G}_1$ of $\mathcal{G}$ (*cf*. Definition 1.4.5.2.1) (which is the smallest normal subgroup of $\mathcal{G}$ such that the factor group is Abelian) is contained in $\mathcal{M}$. Hence all maximal subgroups of index 2 in $\mathcal{G}$ contain $\mathcal{G}_1$. If one defines $\mathcal{N} := \cap\{\mathcal{M} \leq \mathcal{G} \mid [\mathcal{G} : \mathcal{M}] = 2\}$, then $\mathcal{G}/\mathcal{N}$ is an elementary Abelian 2-group and hence a vector space over the field with two elements. The maximal subgroups of $\mathcal{G}/\mathcal{N}$ are the maximal subspaces of this vector space, hence their number is $2^a - 1$, where $a := \dim_{\mathbb{Z}/2\mathbb{Z}}(\mathcal{G}/\mathcal{N})$.

This shows the following:

Corollary 1.4.6.1.2. The number of subgroups of $\mathcal{G}$ of index 2 is of the form $2^a - 1$ for some $a \geq 0$. □

Dealing with subgroups of index 3, one has the following:

Proposition 1.4.6.1.3. Let $\mathcal{U}$ be a subgroup of the group $\mathcal{G}$ with $[\mathcal{G} : \mathcal{U}] = 3$. Then $\mathcal{U}$ is either a normal subgroup of $\mathcal{G}$ or $\mathcal{G}/\mathrm{core}(\mathcal{U}) \cong \mathcal{S}_3$ and there are three subgroups of $\mathcal{G}$ conjugate to $\mathcal{U}$. □

Proof. $\mathcal{G}/\mathrm{core}\,(\mathcal{U})$ is isomorphic to a subgroup of $\mathcal{S}ym_3$ that acts primitively on $\{1, 2, 3\}$. Hence either $\mathcal{G}/\mathrm{core}\,(\mathcal{U}) \cong \mathcal{C}yc_3$ and $\mathcal{U} = \mathrm{core}\,(\mathcal{U})$ is a normal subgroup of $\mathcal{G}$ or $\mathcal{G}/\mathrm{core}\,(\mathcal{U}) \cong \mathcal{S}ym_3$, $\mathcal{U}/\mathrm{core}\,(\mathcal{U}) \cong \mathcal{C}yc_2$ and there are three subgroups of $\mathcal{G}$ conjugate to $\mathcal{U}$. QED

1.4.6.2. Three-dimensional space groups

We now come to space groups. By Lemma 1.4.5.2.3, all three-dimensional space groups are soluble. Theorem 1.4.5.3.1 says that the index of a maximal subgroup of a soluble group is a prime power (or infinite). Since the index of a maximal subgroup of a space group is always finite (see Corollary 1.4.4.2.7), we get:

Corollary 1.4.6.2.1. Let $\mathcal{G}$ be a three-dimensional space group and $\mathcal{M} \leq \mathcal{G}$ a maximal subgroup. Then $[\mathcal{G} : \mathcal{M}]$ is a prime power. □

Let $\mathcal{R}$ be a three-dimensional space group and $\mathcal{P} = \mathcal{R}/\mathcal{T}(\mathcal{R})$ its point group. It is well known that the order of $\mathcal{P}$ is of the form $2^a 3^b$ with $a = 0, 1, 2, 3$ or 4 and $b = 0, 1$. By Corollary 1.4.4.2.4, the *t*-subgroups of $\mathcal{R}$ are in one-to-one correspondence with the subgroups of $\mathcal{P}$. Let us look at the *t*-subgroups of $\mathcal{R}$ of index 3. It is clear that $\mathcal{P}$ has no subgroup of index 3 if $b = 0$, since the index of a subgroup divides the order of the finite group $\mathcal{P}$ by the theorem of Lagrange. If $b = 1$, then any subgroup $\mathcal{S}$ of $\mathcal{P}$ of index 3 has order $|\mathcal{P}|/3 = 2^a$ and hence is a Sylow 2-subgroup of $\mathcal{P}$. Therefore there is such a subgroup $\mathcal{S}$ of index 3 in $\mathcal{P}$ by the first theorem of Sylow, Theorem 1.4.3.3.1. By the second theorem of Sylow, Theorem 1.4.3.3.2, all these Sylow 2-subgroups of $\mathcal{P}$ are

references