The last entry for each point group contains the Symmetry of special projections, i.e. the plane point group that is obtained if the three-dimensional point group is projected along a symmetry direction. The special projection directions are the same as for the space groups; they are listed in Section 2.2.14. The relations between the axes of the three-dimensional point group and those of its two-dimensional projections can easily be derived with the help of the stereographic projection. No projection symmetries are listed for the two-dimensional point groups.

Note that the symmetry of a projection along a certain direction may be higher than the symmetry of the crystal face normal to that direction. For example, in point group \( T \) all faces have face symmetry 1, whereas projections along any direction have symmetry 2; in point group \( 422 \), the faces (001) and (00T) have face symmetry 4, whereas the projection along [001] has symmetry 4/\( m \).

### 10.1.2.2. Crystal and point forms

For a point group \( P \) a crystal form is a set of all symmetrically equivalent faces; a point form is a set of all symmetrically equivalent points. Crystal and point forms in point groups correspond to ‘crystallographic orbits’ in space groups; cf. Section 8.3.2.

Two kinds of crystal and point forms with respect to \( P \) can be distinguished. They are defined as follows:

(i) General form: A face is called ‘general’ if only the identity operation transforms the face onto itself. Each complete set of symmetrically equivalent ‘general faces’ is a general crystal form. The multiplicity of a general form, i.e. the number of its faces, is the order of \( P \). In the stereographic projection, the poles of general faces do not lie on symmetry elements of \( P \).

A point is called ‘general’ if its site symmetry, i.e. the group of symmetry operations that transforms this point onto itself, is \( 1 \). A general point form is a complete set of symmetrically equivalent ‘general points’.

(ii) Special form: A face is called ‘special’ if it is transformed onto itself by at least one symmetry operation of \( P \); in addition to the identity. Each complete set of symmetrically equivalent ‘special faces’ is called a special crystal form. The face symmetry of a special face is the group of symmetry operations that transforms this face onto itself; it is a subgroup of \( P \). The multiplicity of a special crystal form is the multiplicity of the general form divided by the order of the face-symmetry group. In the stereographic projection, the poles of special faces lie on symmetry elements of \( P \). The Miller indices of a special crystal form obey restrictions like \{h00\}, \{hhl\}, \{100\}.

A point is called ‘special’ if its site symmetry is higher than \( 1 \). A special point form is a complete set of symmetrically equivalent ‘special points’. The multiplicity of a special point form is the multiplicity of the general form divided by the order of the site-symmetry group. It is thus the same as that of the corresponding special crystal form. The coordinates of the points of a special point form obey restrictions, like \( x, y, 0; x, x, z; x, 0, 0 \). The point 0, 0, 0 is not considered to be a point form.

In two dimensions, point groups 1, 2, 3, 4 and 6 and in three dimensions, point groups 1 and \( T \) have no special crystal and point forms.

General and special crystal and point forms can be represented by their sets of equivalent Miller indices \{hkl\} and point coordinates \( x, y, z \). Each set of these ‘triplets’ stands for infinitely many crystal forms or point forms which are obtained by independent variation of the values and signs of the Miller indices \( h, k, l \) or the point coordinates \( x, y, z \).

It should be noted that for crystal forms, owing to the well known ‘law of rational indices’, the indices \( h, k, l \) must be integers; no such restrictions apply to the coordinates \( x, y, z \), which can be rational or irrational numbers.

**Example**

In point group 4, the general crystal form \{hkl\} stands for the set of all possible tetragonal pyramids, pointing either upwards or downwards, depending on the sign of \( l \); similarly, the general point form \( x, y, z \) includes all possible squares, lying either above or below the origin, depending on the sign of \( z \). For the limiting cases \( l = 0 \) or \( z = 0 \), see below.

In order to survey the infinite number of possible forms of a point group, they are classified into Wyckoff positions of crystal and point forms, for short Wyckoff positions. This name has been chosen in analogy to the Wyckoff positions of space groups; cf. Sections 2.2.11 and 8.3.2. In point groups, the term ‘position’ can be visualized as the position of the face poles and points in the stereographic projection. Each ‘Wyckoff position’ is labelled by a Wyckoff letter.

**Definition**

A ‘Wyckoff position of crystal and point forms’ consists of all those crystal forms (point forms) of a point group \( P \) for which the face poles (points) are positioned on the same set of conjugate symmetry elements of \( P \); i.e. for each face (point) of one form there is one face (point) of every other form of the same Wyckoff position that has exactly the same face (site) symmetry.

Each point group contains one ‘general Wyckoff position’ comprising all general crystal and point forms. In addition, up to two ‘special Wyckoff positions’ may occur in two dimensions and up to six in three dimensions. They are characterized by the different sets of conjugate face and site symmetries and correspond to the seven positions of a pole in the interior, on the three edges, and at the three vertices of the so-called ‘characteristic triangle’ of the stereographic projection.

**Examples**

(1) All tetragonal pyramids \{hkl\} and tetragonal prisms \{hkl\} in point group 4 have face symmetry 1 and belong to the same general Wyckoff position \( 4b \), with Wyckoff letter \( b \).

(2) All tetragonal pyramids and tetragonal prisms in point group \( 4mn \) belong to two special Wyckoff positions, depending on the orientation of their face-symmetry groups \( m \) with respect to the crystal axes: For the ‘oriented face symmetry’ \( m \), the forms \{011\} and \{100\} belong to Wyckoff position \( 4c \); for the oriented face symmetry \( m \), the forms \{hll\} and \{110\} belong to Wyckoff position \( 4b \). The face symmetries \( m \) and \( m \) are not conjugate in point group \( 4mn \). For the analogous ‘oriented site symmetries’ in space groups, see Section 2.2.12.

It is instructive to subdivide the crystal forms (point forms) of one Wyckoff position further, into characteristic and noncharacteristic forms. For this, one has to consider two symmetries that are connected with each crystal (point) form:

(i) The point group \( P \) by which a form is generated (generating point group), i.e. the point group in which it occurs;

(ii) The full symmetry (inherent symmetry) of a form (considered as a polyhedron by itself), here called eigensymmetry \( C \). The eigensymmetry point group \( C \) is either the generating point group itself or a subgroup of it.

**Examples**

(1) Each tetragonal pyramid \{hkl\} \( l \neq 0 \) of Wyckoff position \( 4b \) in point group 4 has generating symmetry 4 and eigensymmetry...
4mm; each tetragonal prism \(\{hk0\}\) of the same Wyckoff position has generating symmetry 4 again, but eigensymmetry 4/mmm.

(2) A cube \(\{100\}\) may have generating symmetry 23, \(m\bar{3}, 432, \bar{3}m\) or \(m\bar{5}m\), but its eigensymmetry is always \(m\bar{3}m\).

The eigensymmetries and the generating symmetries of the 47 crystal forms (point forms) are listed in Table 10.1.2.3. With the help of this table, one can find the various point groups in which a given crystal form (point form) occurs, as well as the face (site) symmetries that it exhibits in these point groups; for experimental methods see Sections 10.2.2 and 10.2.3.

With the help of the two groups \(\mathcal{P}\) and \(\mathcal{C}\), each crystal or point form occurring in a particular point group can be assigned to one of the following two categories:

(i) characteristic form, if eigensymmetry \(\mathcal{C}\) and generating symmetry \(\mathcal{P}\) are the same;

(ii) noncharacteristic form, if \(\mathcal{C}\) is a proper supergroup of \(\mathcal{P}\).

The importance of this classification will be apparent from the following examples.

**Examples**

(1) A pedion and a pinacoid are noncharacteristic forms in all crystallographic point groups in which they occur.

(2) all other crystal or point forms occur as characteristic forms in their eigensymmetry group \(\mathcal{C}\).

(3) a tetragonal pyramid is noncharacteristic in point group 4 and characteristic in 4mm;

(4) a hexagonal prism can occur in nine point groups (12 Wyckoff positions) as a noncharacteristic form; in \(6/mmm\), it occurs in two Wyckoff positions as a characteristic form.

The general forms of the 13 point groups with no, or only one, symmetry direction (‘monoxial groups’) 1, 2, 3, 4, 6, \(\bar{1}, m, 3, 4, 6 = \bar{3}/m, 2/m, \bar{4}/m, \bar{6}/m\) are always noncharacteristic, i.e. their eigensymmetries are enhanced in comparison with the generating point groups. The general positions of the other 19 point groups always contain characteristic crystal forms that may be used to determine the point group of a crystal uniquely (cf. Section 10.2.2).

So far, we have considered the occurrence of one crystal or point form in different point groups and different Wyckoff positions. We now turn to the occurrence of different kinds of crystal or point forms in one and the same Wyckoff position of a particular point group.

In a Wyckoff position, crystal forms (point forms) of different eigensymmetries may occur; the crystal forms (point forms) with the lowest eigensymmetry (which is always well defined) are called basic forms (German: Grundformen) of that Wyckoff position. The crystal and point forms of higher eigensymmetry are called limiting forms (German: Grenzformen) (cf. Table 10.1.2.3). These forms are always noncharacteristic.

Limiting forms\(\dagger\) occur for certain restricted values of the Miller indices or point coordinates. They always have the same multiplicity and oriented face (site) symmetry as the corresponding basic forms because they belong to the same Wyckoff position. The enhanced eigensymmetry of a limiting form may or may not be accompanied by a change in the topology\(\ddagger\) of its polyhedra, compared with that of a basic form. In every case, however, the name of a limiting form is different from that of a basic form.

The face poles (or points) of a limiting form lie on symmetry elements of a supergroup of the point group that are not symmetry elements of the point group itself. There may be several such supergroups.

**Examples**

(1) In point group 4, the (noncharacteristic) crystal forms \(\{hkl\}\) \((l \neq 0)\) (tetragonal pyramids) of eigensymmetry 4mm are basic forms of the general Wyckoff position 4h, whereas the forms \(\{hk0\}\) (tetragonal prisms) of higher eigensymmetry 4/mmm are ‘limiting general forms’. The face poles of forms \(\{hk0\}\) lie on the horizontal mirror plane of the supergroup 4/m.\(\dagger\)

(2) In point group 4mm, the (characteristic) special crystal forms \(\{hkl\}\) with eigensymmetry 4mm are ‘basic forms’ of the special Wyckoff position 4c, whereas \(\{100\}\) with eigensymmetry 4/mmm is a ‘limiting special form’. The face poles of \(\{100\}\) are located on the intersections of the vertical mirror planes of the point group 4mm with the horizontal mirror plane of the supergroup 4/mmm, i.e. on twofold axes of 4/mmm.

Whereas basic and limiting forms belonging to one ‘Wyckoff position’ are always clearly distinguished, closer inspection shows that a Wyckoff position may contain different ‘types’ of limiting forms. We need, therefore, a further criterion to classify the limiting forms of one Wyckoff position into types: A type of limiting form of a Wyckoff position consists of all those limiting forms for which the face poles (points) are located on the same set of additional conjugate symmetry elements of the holohedral point group (for the trigonal point groups, the hexagonal holohedry 6/mmm has to be taken). Different types of limiting forms may have the same eigensymmetry and the same topology, as shown by the examples below. The occurrence of two topologically different polyhedra as two ‘realizations’ of one type of limiting form in point groups 23, \(m\bar{3}m\) and 432 is explained below in Section 10.1.2.4, Notes on crystal and point forms, item (viii).

**Examples**

(1) In point group 32, the limiting general crystal forms are of four types:

(i) ditrigonal prisms, eigensymmetry \(\bar{6}2m\) (face poles on horizontal mirror plane of holohedry 6/mmm);

(ii) trigonal dipyrmaids, eigensymmetry \(\bar{6}2m\) (face poles on one kind of vertical mirror plane);

(iii) rhombohedra, eigensymmetry \(\bar{5}m\) (face poles on second kind of vertical mirror plane);

(iv) hexagonal prisms, eigensymmetry 6/mmm (face poles on horizontal twofold axes).

Types (i) and (ii) have the same eigensymmetry but different topologies; types (i) and (iv) have the same topology but different eigensymmetries; type (iii) differs from the other three types in both eigensymmetry and topology.

(2) In point group 222, the face poles of the three types of general limiting forms, rhombic prisms, are located on the three (non-equivalent) symmetry planes of the holohedry mmm. Geometrically, the axes of the prisms are directed along the three non-equivalent orthorhombic symmetry directions. The three types

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\* For a survey of these relations, as well as of the ‘limiting forms’, it is helpful to consider the (seven) normalizers of the crystallographic point groups in the group of all rotations and reflections (orthogonal group, sphere group); normalizers of the crystallographic and noncrystallographic point groups are listed in Tables 15.4.1.1 and 15.4.1.2.

\(\dagger\) The treatment of ‘limiting forms’ in the literature is quite ambiguous. In some textbooks, limiting forms are omitted or treated as special forms in their own right; other authors define only limiting general forms and consider limiting special forms as if they were new special forms. For additional reading, see P. Niggli (1941), pp. 80–98.

\(\ddagger\) The topology of a polyhedron is determined by the numbers of its vertices, edges and faces, by the number of vertices of each face and by the number of faces meeting in each vertex.
of limiting forms have the same eigensymmetry and the same topology but different orientations.

Similar cases occur in point groups 422 and 622 (cf. Table 10.1.2.3, footnote *).

Not considered in this volume are limiting forms of another kind, namely those that require either special metrical conditions for the axial ratios or irrational indices or coordinates (which always can be closely approximated by rational values). For instance, a rhombohedral can, for special axial ratios, appear as a tetragonal or even as a cubic tetrahedron; similarly, a rhombohedron can degenerate to a cube. For special irrational indices, a ditetragonal prism changes to a (noncrystallographic) octagonal prism, a rhombohedral to a rhombohedral pyramid or a crystallographic pentagon-dodecahedron to a regular polygon-dodecahedron. These kinds of limiting forms are listed by A. Niggli (1963).

In conclusion, each general or special Wyckoff position always contains one set of basic crystal (point) forms. In addition, it may contain one or more sets of limiting forms of different types. As a rule, each type comprises polyhedra of the same eigensymmetry and topology and, hence, of the same name, for instance 'ditetragonal pyramid'. The name of the basic general form is often used to designate the corresponding crystal class, for instance 'ditetragonal-pyramidal class'; some of these names are listed in Table 10.1.2.4.

10.1.2.3. Description of crystal and point forms

The main part of each point-group table describes the general and special crystal and point forms of that point group, in a manner analogous to the positions in a space group. The general Wyckoff position is given at the top, followed downwards by the special Wyckoff positions with decreasing multiplicity. Within each Wyckoff position, the first block refers to the basic forms, the subsequent blocks list the various types of limiting form, if any.

The columns, from left to right, contain the following data (further details are to be found below in Section 10.1.2.4, Notes on crystal and point forms):

- Column 1: Multiplicity of the 'Wyckoff position', i.e. the number of equivalent faces and points of a crystal or point form.
- Column 2: Wyckoff letter. Each general or special 'Wyckoff position' is designated by a 'Wyckoff letter', analogous to the Wyckoff letter of a position in a space group (cf. Section 2.2.11).
- Column 3: Face symmetry or site symmetry, given in the form of an 'oriented point-group symbol', analogous to the oriented site-symmetry symbols of space groups (cf. Section 2.2.12). The face symmetry is also the symmetry of etch pits, striations and other face markings. For the two-dimensional point groups, this column contains the edge symmetry, which can be either $1$ or $m$.
- Column 4: Name of crystal form. If more than one name is in common use, several are listed. The names of the limiting forms are also given. The crystal forms, their names, eigensymmetries and occurrence in the point groups are summarized in Table 10.1.2.3, which may be useful for determinative purposes, as explained in Sections 10.2.2 and 10.2.3. There are 47 different types of crystal form. Frequently, 48 are quoted because 'sphenoïd' and 'dome' are considered as two different forms. It is customary, however, to regard them as the same form, with the name 'dihedron'.

Name of point form (printed in italics). There exists no general convention on the names of the point forms. Here, only one name is given, which does not always agree with that of other authors. The names of the point forms are also contained in Table 10.1.2.3. Note that the same point form, 'line segment', corresponds to both sphenoïd and dome. The letter in parentheses after each name of a point form is explained below.

Column 5: Miller indices ($hkli$) for the symmetrically equivalent faces (edges) of a crystal form. In the trigonal and hexagonal crystal systems, when referring to hexagonal axes, Bravais–Miller indices ($hkli$) are used, with $h + k + l = 0$.

Coordinates $x, y, z$ for the symmetrically equivalent points of a point form are not listed explicitly because they can be obtained from data in this volume as follows: after the name of the point form, a letter is given in parentheses. This is the Wyckoff letter of the corresponding position in the symmorphic $P$ space group that belongs to the point group under consideration. The coordinate triplets of this (general or special) position apply to the point form of the point group.

The triplets of Miller indices ($hkli$) and point coordinates $x, y, z$ are arranged in such a way as to show analogous sequences; they are both based on the same set of generators, as described in Sections 2.2.10 and 8.3.5. For all point groups, except those referred to a hexagonal coordinate system, the correspondence between the ($hkli$) and the $x, y, z$ triplets is immediately obvious.\footnote{The matrices of corresponding triplets ($hkli$) and $x, y, z$, i.e. of triplets generated by the same symmetry operation from ($hkli$) and $x, y, z$, are inverse to each other, provided the $x, y, z$ and $x, y, z$ are regarded as columns and the ($hkli$) and ($hkli$) as rows: this is due to the contravariant and covariant nature of the point coordinates and Miller indices, respectively. Note that for orthogonal matrices the inverse matrix equals the transposed matrix; in crystallography, this applies to all coordinate systems (including the rhombohedral one), except for the hexagonal system. The matrices for the symmetry operations occurring in the crystallographic point groups are listed in Tables 11.2.2.1 and 11.2.2.2.}

The sets of symmetrically equivalent crystal faces also represent the sets of equivalent reciprocal-lattice points, as well as the sets of equivalent X-ray (neutron) reflections.

Examples

1. In point group $\overline{4}$, the general crystal form $4b$ is listed as $(hkli)\overline{hkli}\overline{hkli}$: the corresponding general position $4h$ of the symmetric space group $P\overline{4}$ reads $x, y, z; \overline{x}, \overline{y}, \overline{z}; y, x, z; \overline{y}, x, \overline{z}$.

2. In point group 3, the general crystal form $3b$ is listed as $(hkli)$ (or $hklm$) with $i = -(h + k)$; the corresponding general position $3d$ of the symmetric space group $P3$ reads $x, y, z; \overline{y}, x, z; -x + y, z$.

3. The Miller indices of the cubic point groups are arranged in one, two or four blocks of $(3 \times 4)$ entries. The first block (upper left) belongs to point group 23. The second block (upper right) belongs to the diagonal twofold axes in $342$ and $m\overline{3}m$ or to the diagonal mirror plane in $\overline{4}3m$. In point groups $m\overline{3}$ and $\overline{m}\overline{3}$, the lower one or two blocks are derived from the upper blocks by application of the inversion.

10.1.2.4. Notes on crystal and point forms

(i) As mentioned in Section 10.1.2.2, each set of Miller indices of a given point group represents infinitely many face forms with the same name. Exceptions occur for the following cases.

Some special crystal forms occur with only one representative. Examples are the pinacoid {001}, the hexagonal prism {1010} and the cube {100}. The Miller indices of these forms consist of fixed numbers and signs and contain no variables.

In a few noncentrosymmetric point groups, a special crystal form is realized by two representatives: they are related by a centre of symmetry that is not part of the point-group symmetry. These cases are listed as two pedions (001) and (001).

\footnote{For the exceptions in the cubic crystal system cf. Section 10.1.2.4, Notes on crystal and point forms, item (viii)}