10.1. CRYSTALLOGRAPHIC AND NONCRYSTALLOGRAPHIC POINT GROUPS

(2) In point group 32, the general form is a trigonal trapezohedron \( \{hkl\} \); this form can be considered as two opposite trigonal pyramids, rotated with respect to each other by an angle \( \chi \). The trapezohedron changes into the limiting forms ‘trigonal dipyramid’ \( \{hkl\} \) for \( \chi = 0^\circ \) and ‘rhombohedron’ \( \{hkl\} \) for \( \chi = 60^\circ \).

(vii) One and the same type of polyhedron can occur as a general, special or limiting form.

Examples
(1) A tetragonal dipyramid is a general form in point group 4/m, a special form in point group 4/mmm and a limiting general form in point groups 422 and 42m.

(2) A tetragonal prism appears in point group 42m both as a basic special form (4b) and as a limiting special form (4c).

(viii) A peculiarity occurs for the cubic point groups. Here the crystal forms \( \{hkl\} \) are realized as two topologically different kinds of polyhedra with the same face symmetry, multiplicity and, in addition, the same eigensymmetry. The realization of one or other of these forms depends upon whether the Miller indices obey the conditions \(|h| > |l| \) or \(|h| < |l|\), i.e. whether, in the stereographic projection, a face pole is located between the directions \([110]\) and \([111]\) or between the directions \([111]\) and \([001]\). These two kinds of polyhedra have to be considered as two realizations of one type of crystal form because their face poles are located on the same set of conjugate symmetry elements. Similar considerations apply to the point forms \(x, x, z\).

In the point groups \(m\overline{5}m\) and \(\overline{4}3m\), the two kinds of polyhedra represent two realizations of one special ‘Wyckoff position’; hence, they have the same Wyckoff letter. In the groups 23, \(m\overline{5}\) and 432, they represent two realizations of the same type of limiting formal forms. In the tables of the cubic point groups, the two entries are always connected by braces.

The same kind of peculiarity occurs for the twoicosahedral point groups, as mentioned in Section 10.1.4 and listed in Table 10.1.4.3.

10.1.2.5. Names and symbols of the crystal classes

Several different sets of names have been devised for the 32 crystal classes. Their use, however, has greatly declined since the introduction of the international point-group symbols. As examples, two sets (both translated into English) that are frequently found in the literature are given in Table 10.1.2.4. To the name of the class the name of the system has to be added; e.g. ‘tetragonal pyramidal’ or ‘tetragonal tetartohedry’.

Note that Friedel (1926) based his nomenclature on the point symmetry of the lattice. Hence, two names are given for the five trigonal point groups, depending whether the lattice is hexagonal or rhombohedral: e.g. ‘hexagonal ogdohedry’ and ‘rhombohedral tetartohedry’.

10.1.3. Subgroups and supergroups of the crystallographic point groups

In this section, the sub- and supergroup relations between the crystallographic point groups are presented in the form of a ‘family tree’. Figs. 10.1.3.1 and 10.1.3.2 apply to two and three dimensions. The sub- and supergroup relations between two groups are represented by solid or dashed lines. For a given point group \(P\) of order \(k_P\) the lines to groups of lower order connect \(P\) with all its maximal subgroups \(H\) with orders \(k_H\); the index \([i]\) of each subgroup is given by the ratio of the orders \(k_P/k_H\). The lines to groups of higher order connect \(P\) with all its minimal supergroups \(S\)

Fig. 10.1.3.1. Maximal subgroups and minimal supergroups of the two-dimensional crystallographic point groups. Solid lines indicate maximal normal subgroups; double solid lines mean that there are two maximal normal subgroups with the same symbol. Dashed lines refer to sets of maximal conjugate subgroups. The group orders are given on the left.

with orders \(k_S\); the index \([i]\) of each supergroup is given by the ratio \(k_S/k_P\). In other words: if the diagram is read downwards, subgroup relations are displayed; if it is read upwards, supergroup relations are revealed. The index is always an integer (theorem of Lagrange) and can be easily obtained from the group orders given on the left of the diagrams. The highest index of a maximal subgroup is \([3]\) for two dimensions and \([4]\) for three dimensions.

Two important kinds of subgroups, namely sets of conjugate subgroups and normal subgroups, are distinguished by dashed and solid lines. They are characterized as follows:

The subgroups \(H_1, H_2, \ldots, H_n\) of a group \(P\) are conjugate subgroups if \(H_1, H_2, \ldots, H_n\) are symmetrically equivalent in \(P\), i.e. if for every pair \(H_i, H_j\) at least one symmetry operation \(W\) of \(P\) exists which maps \(H_i\) onto \(H_j: W^{-1}WH = H_j\); cf. Section 8.3.6.

Examples
(1) Point group 3m has three different mirror planes which are equivalent due to the threefold axis. In each of the three maximal subgroups of type \(m\), one of these mirror planes is retained. Hence, the three groups \(m\) are conjugate in 3m. This set of conjugate subgroups is represented by one dashed line in Figs. 10.1.3.1 and 10.1.3.2.

(2) Similarly, group 432 has three maximal conjugate subgroups of type 422 and four maximal conjugate subgroups of type 32.

The subgroup \(H\) of a group \(P\) is a normal (or invariant) subgroup if no subgroup \(H'\) of \(P\) exists that is conjugate to \(H\) in \(P\). Note that this does not imply that \(H\) is also a normal subgroup of any supergroup of \(P\). Subgroups of index \([2]\) are always normal and maximal. (The role of normal subgroups for the structure of space groups is discussed in Section 8.1.6.)

Examples
(1) Fig. 10.1.3.2 shows two solid lines between point groups 422 and 222, indicating that 422 has two maximal normal subgroups 222 of index \([2]\). The symmetry elements of one subgroup are rotated by \(45^\circ\) around the \(c\) axis with respect to those of the other subgroup. Thus, in one subgroup the symmetry elements of the two secondary, in the other those of the two tertiary tetragonal symmetry directions (cf. Table 2.2.4.1) are retained, whereas the primary twofold axis is the same for both subgroups. There exists no symmetry operation of 422 that maps one subgroup onto the other. This is illustrated by the stereograms below. The two normal subgroups can be indicated by the ‘oriented