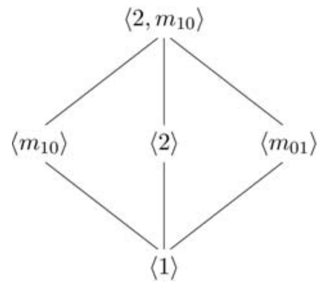

Figure 1.1.3.2
Subgroup diagram for the symmetry group $2mm$ of a rectangle.

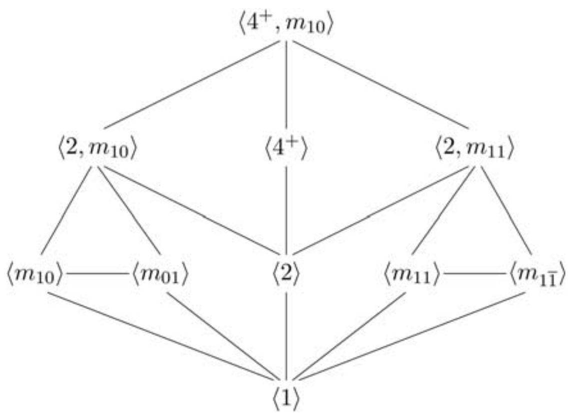

Figure 1.1.3.3
Subgroup diagram for the symmetry group $4mm$ of the square.

(vii) In the symmetry group $4mm$ of the square, the reflections m_{10} and m_{01} together with their product 2 and the identity element 1 form a subgroup of order 4. This subgroup can be recognized in the subgroup diagram of $4mm$ as the subdiagram of the subgroups of $\langle 2, m_{10}\rangle$ in the left part of Fig. 1.1.3.3 which coincides with the subgroup diagram of $2mm$ in Fig. 1.1.3.2. A different subgroup of order 4 is formed by the other pair of perpendicular reflections m_{11}, $m_{1\bar{1}}$ together with 2 and 1 and a third subgroup of order 4 is the cyclic subgroup $\langle 4^+\rangle$ generated by the fourfold rotation (see Fig. 1.1.3.3).

1.1.4. Cosets

A subgroup allows us to partition a group into disjoint subsets of the same size, called *cosets*.

Definition. Let $\mathcal{H} = \{h_1, h_2, h_3, \ldots\}$ be a subgroup of $\mathcal{G}$. Then for $g \in \mathcal{G}$ the set

$$g\mathcal{H} := \{gh_1, gh_2, gh_3, \ldots\} = \{gh \mid h \in \mathcal{H}\}$$

is called the *left coset* of $\mathcal{H}$ with *representative* g. Analogously, the *right coset* with representative g is defined as

$$\mathcal{H}g := \{h_1 g, h_2 g, h_3 g, \ldots\} = \{hg \mid h \in \mathcal{H}\}.$$

The coset $e\mathcal{H} = \mathcal{H} = \mathcal{H}e$ is called the *trivial coset* of $\mathcal{H}$.

Remarks

(i) Since two elements gh and gh' in the same coset $g\mathcal{H}$ can only be the same if $h = h'$, the elements of $g\mathcal{H}$ are in one-to-one correspondence with the elements of $\mathcal{H}$. In particular, for a finite subgroup $\mathcal{H}$ the number of elements in each coset of $\mathcal{H}$ equals the order $|\mathcal{H}|$ of the subgroup $\mathcal{H}$.

(ii) Every element contained in $g\mathcal{H}$ may serve as representative for this coset, *i.e.* $g'\mathcal{H} = g\mathcal{H}$ for every $g' \in g\mathcal{H}$. In particular, if an element g'' is contained in the intersection $g\mathcal{H} \cap g'\mathcal{H}$ of two cosets, one has $g''\mathcal{H} = g\mathcal{H}$ and $g''\mathcal{H} = g'\mathcal{H}$. This implies that two cosets are either disjoint (*i.e.* contain no common element) or they are equal.

These two remarks have an important consequence: since an element $g \in \mathcal{G}$ is contained in the coset $g\mathcal{H}$, the cosets of $\mathcal{H}$ partition the elements of $\mathcal{G}$ into sets of the same cardinality as $\mathcal{H}$ (which is of the order of $\mathcal{H}$ in the case where this is finite).

Definition. If the number of different cosets of a subgroup $\mathcal{H} \le \mathcal{G}$ is finite, this number is called the *index* of $\mathcal{H}$ in $\mathcal{G}$, denoted by $[i]$ or $[\mathcal{G} : \mathcal{H}]$. Otherwise, $\mathcal{H}$ is said to have *infinite index* in $\mathcal{G}$.

In the case of a finite group, the partitioning of the elements of $\mathcal{G}$ into the cosets of $\mathcal{H}$ shows that both the order of $\mathcal{H}$ and the index of $\mathcal{H}$ in $\mathcal{G}$ divide the order of $\mathcal{G}$. This is summarized in the following famous result.

Lagrange's theorem

For a finite group $\mathcal{G}$ and a subgroup $\mathcal{H}$ of $\mathcal{G}$ one has

$$|\mathcal{G}| = |\mathcal{H}| \cdot [\mathcal{G} : \mathcal{H}],$$

i.e. the order of a subgroup multiplied by its index gives the order of the full group.

For example, a group of order n cannot have a proper subgroup of order larger than $n/2$.

Whether or not two cosets of a subgroup $\mathcal{H}$ are equal depends on whether the quotient of their representatives is contained in $\mathcal{H}$: for left cosets one has $g\mathcal{H} = g'\mathcal{H}$ if and only if $g^{-1}g' \in \mathcal{H}$ and for right cosets $\mathcal{H}g = \mathcal{H}g'$ if and only if $g'g^{-1} \in \mathcal{H}$.

Definition. If $\mathcal{H}$ is a subgroup of $\mathcal{G}$ and $g_1, g_2, g_3, \ldots \in \mathcal{G}$ are such that $g_i\mathcal{H} \ne g_j\mathcal{H}$ for $i \ne j$, and every $g \in \mathcal{G}$ is contained in some left coset $g_i\mathcal{H}$, then $g_1, g_2, g_3, \ldots$ is called a system of *left coset representatives* of $\mathcal{G}$ relative to $\mathcal{H}$. It is customary to choose $g_1 = e$ so that the coset $g_1\mathcal{H} = e\mathcal{H} = \mathcal{H}$ is the subgroup $\mathcal{H}$ itself. The decomposition

$$\mathcal{G} = \mathcal{H} \cup g_2\mathcal{H} \cup g_3\mathcal{H} \ldots$$

is called the *coset decomposition* of $\mathcal{G}$ into left cosets relative to $\mathcal{H}$.

Analogously, $g_1', g_2', g_3', \ldots \in \mathcal{G}$ is called a system of *right coset representatives* if $\mathcal{H}g_i' \ne \mathcal{H}g_j'$ for $i \ne j$ and every $g \in \mathcal{G}$ is contained in some right coset $\mathcal{H}g_i$. Again, one usually chooses $g_1' = e$ and the decomposition

$$\mathcal{G} = \mathcal{H} \cup \mathcal{H}g_2' \cup \mathcal{H}g_3' \ldots$$

is called the *coset decomposition* of $\mathcal{G}$ into right cosets relative to $\mathcal{H}$.

To obtain the coset decomposition one starts by choosing $\mathcal{H}$ as the first coset (with representative e). Next, an element $g_2 \in \mathcal{G}$ with $g_2 \notin \mathcal{H}$ is selected as representative for the second coset $g_2\mathcal{H}$. For the third coset, an element $g_3 \in \mathcal{G}$ with $g_3 \notin \mathcal{H}$ and $g_3 \notin g_2\mathcal{H}$ is required. If at a certain stage the cosets $\mathcal{H}, g_2\mathcal{H}, \ldots, g_m\mathcal{H}$ have been defined but do not yet exhaust $\mathcal{G}$, an element g_{m+1} not contained in the union $\mathcal{H} \cup g_2\mathcal{H} \cup \ldots \cup g_m\mathcal{H}$ is chosen as representative for the next coset.

Examples

(i) Let $\mathcal{G} = 3m$ be the symmetry group of an equilateral triangle and $\mathcal{H} = \langle 3^+\rangle$ its subgroup containing the rotations. Then for every reflection $m \in \mathcal{G}$ the elements e, m form a system of coset representatives of $\mathcal{G}$ relative to $\mathcal{H}$ and the coset decomposition is $\mathcal{G} = \{1, 3^+, 3^-\} \cup \{m_{10}, m_{01}, m_{11}\}$.