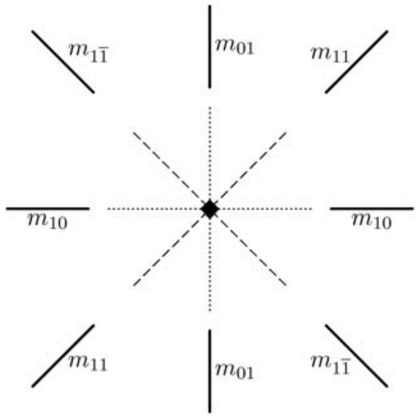

Figure 1.1.5.1
Symmetry group of an eightfold star.

$(8 + 12\mathbb{Z}) + (9 + 12\mathbb{Z}) = 17 + 12\mathbb{Z} = 5 + 12\mathbb{Z}$. In the factor group $\mathbb{Z}/12\mathbb{Z}$, the clock is imagined as a circle of circumference 12 around which the line of integers is wrapped so that integers with a difference of 12 are located at the same position on the circle.

The clock example is a special case of factor groups of the integers. We have already seen that the set $n\mathbb{Z} = \{na \mid a \in \mathbb{Z}\}$ of multiples of a natural number n forms a subgroup of index n in $\mathbb{Z}$. This is a normal subgroup, since $\mathbb{Z}$ is an abelian group. The factor group $\mathbb{Z}/n\mathbb{Z}$ represents the addition of integers *modulo n*.

Examples

(i) If we take $\mathcal{G}$ to be the symmetry group $4mm$ of the square and choose as normal subgroup the subgroup $\mathcal{H} = \langle 4^+ \rangle$ generated by the fourfold rotation, we obtain a factor group $\mathcal{G}/\mathcal{H}$ with two elements, namely the cosets $\mathcal{H} = \{1, 2, 4^+, 4^-\}$ and $m_{10}\mathcal{H} = \{m_{10}, m_{01}, m_{11}, m_{1\bar{1}}\}$. The trivial coset $\mathcal{H}$ is the identity element in the factor group $\mathcal{G}/\mathcal{H}$ and contains the rotations in $4mm$. The other element $m_{10}\mathcal{H}$ in the factor group $\mathcal{G}/\mathcal{H}$ consists of the reflections in $4mm$.
In this example, the separation of the rotations and reflections in $4mm$ into the two cosets $\mathcal{H}$ and $m_{10}\mathcal{H}$ makes it easy to see that the product of two cosets is independent of the chosen representative of the coset: the product of two rotations is again a rotation, hence $\mathcal{H} \cdot \mathcal{H} = \mathcal{H}$, the product of a rotation and a reflection is a reflection, hence $\mathcal{H} \cdot m_{10}\mathcal{H} = m_{10}\mathcal{H} \cdot \mathcal{H} = m_{10}\mathcal{H}$, and finally the product of two reflections is a rotation, hence $m_{10}\mathcal{H} \cdot m_{10}\mathcal{H} = \mathcal{H}$. The multiplication table of the factor group is thus

	$\mathcal{H}$	$m_{10}\mathcal{H}$
$\mathcal{H}$	$\mathcal{H}$	$m_{10}\mathcal{H}$
$m_{10}\mathcal{H}$	$m_{10}\mathcal{H}$	$\mathcal{H}$

(ii) The symmetry group of a square is the same as the symmetry group of an eightfold star, as shown in Fig. 1.1.5.1. If we regard the star as being built from four lines (two dotted and two dashed), then the twofold rotation does not move any of the lines, it only interchanges the points within each line (symmetric with respect to the centre). Regarding the lines as sets of points, the twofold rotation thus does not change anything. The effects of the different symmetry operations on the lines of the eightfold star are then precisely given by the factor group $\mathcal{G}/\mathcal{H}$, where $\mathcal{G}$ is the symmetry group $4mm$ of the square and $\mathcal{H}$ is the normal subgroup generated by the twofold rotation 2: the cosets relative to $\mathcal{H}$ are $\{1, 2\}$, $\{4^+, 4^-\}$, $\{m_{10}, m_{01}\}$, $\{m_{11}, m_{1\bar{1}}\}$, and these cosets collect together the elements of $4mm$ that have the same effect on the lines of the eightfold star. For example, both 4^+ and 4^- interchange both the two dotted and the two dashed lines, m_{10} and m_{01} both interchange the two dashed lines but fix the two dotted lines and m_{11} and $m_{1\bar{1}}$ both interchange the two dotted lines but fix the two dashed lines. Owing to the fact that $\mathcal{H}$ is a normal subgroup, the product of elements from two cosets always lies in the same coset, independent of which elements are chosen from the two cosets. For example, the product of an element from the coset $\{4^+, 4^-\}$ with an element of the coset $\{m_{10}, m_{01}\}$ always gives an element of the coset $\{m_{11}, m_{1\bar{1}}\}$. Working out the products for all pairs of cosets, one obtains the following multiplication table for the factor group $\mathcal{G}/\mathcal{H}$:

	$\{1, 2\}$	$\{4^+, 4^-\}$	$\{m_{10}, m_{01}\}$	$\{m_{11}, m_{1\bar{1}}\}$
$\{1, 2\}$	$\{1, 2\}$	$\{4^+, 4^-\}$	$\{m_{10}, m_{01}\}$	$\{m_{11}, m_{1\bar{1}}\}$
$\{4^+, 4^-\}$	$\{4^+, 4^-\}$	$\{1, 2\}$	$\{m_{11}, m_{1\bar{1}}\}$	$\{m_{10}, m_{01}\}$
$\{m_{10}, m_{01}\}$	$\{m_{10}, m_{01}\}$	$\{m_{11}, m_{1\bar{1}}\}$	$\{1, 2\}$	$\{4^+, 4^-\}$
$\{m_{11}, m_{1\bar{1}}\}$	$\{m_{11}, m_{1\bar{1}}\}$	$\{m_{10}, m_{01}\}$	$\{4^+, 4^-\}$	$\{1, 2\}$

(iii) If one takes cosets with respect to a subgroup that is not normal, the products of elements from two cosets do not lie in a single coset. As we have seen, the left cosets of the group $3m$ of an equilateral triangle with respect to the non-normal subgroup $\mathcal{H} = \{1, m_{10}\}$ are $\{1, m_{10}\}$, $\{3^+, m_{11}\}$ and $\{3^-, m_{01}\}$. Taking products from elements of the first and second coset, we get $1 \cdot 3^+ = 3^+$ and $1 \cdot m_{11} = m_{11}$, which are both in the second coset, but $m_{10} \cdot 3^+ = m_{01}$ and $m_{10} \cdot m_{11} = 3^-$, which are both in the third coset.

1.1.6. Homomorphisms, isomorphisms

In order to relate two groups, mappings between the groups that are compatible with the group operations are very useful.

Recall that a *mapping* φ from a set A to a set B associates to each $a \in A$ an element $b \in B$, denoted by $\varphi(a)$ and called the *image* of a (under φ).

Definition. For two groups $\mathcal{G}$ and $\mathcal{H}$, a mapping φ from $\mathcal{G}$ to $\mathcal{H}$ is called a *group homomorphism* or *homomorphism* for short, if it is compatible with the group operations in $\mathcal{G}$ and $\mathcal{H}$, *i.e.* if

$$\varphi(gg') = \varphi(g)\varphi(g') \text{ for all } g, g' \text{ in } \mathcal{G}.$$

The compatibility with the group operation is captured in the phrase

The image of the product is equal to the product of the images.

Fig. 1.1.6.1 gives a schematic description of the definition of a homomorphism. For φ to be a homomorphism, the two curved arrows are required to give the same result, *i.e.* first multiplying two elements in $\mathcal{G}$ and then mapping the product to $\mathcal{H}$ must be the same as first mapping the elements to $\mathcal{H}$ and then multiplying them.

It follows from the definition of a homomorphism that the identity element of $\mathcal{G}$ must be mapped to the identity element of $\mathcal{H}$ and that the inverse g^{-1} of an element $g \in \mathcal{G}$ must be mapped to the inverse of the image of g, *i.e.* that $\varphi(g^{-1}) = \varphi(g)^{-1}$. In general, however, other elements than the identity element may also be mapped to the identity element of $\mathcal{H}$.