The number of conjugate subgroups of $\mathcal{H}$ in $\mathcal{G}$ is equal to the index of $\mathcal{N}_{\mathcal{G}}(\mathcal{H})$ in $\mathcal{G}$. According to the orbit–stabilizer theorem, the different conjugate subgroups of $\mathcal{H}$ are obtained by conjugating $\mathcal{H}$ with coset representatives for the cosets of $\mathcal{G}$ relative to $\mathcal{N}_{\mathcal{G}}(\mathcal{H})$.

Examples

(i) In an abelian group $\mathcal{G}$, every element is only conjugate with itself, since $ghg^{-1} = h$ for all g, h in $\mathcal{G}$. Therefore each conjugacy class consists of just a single element.

Also, every subgroup $\mathcal{H}$ of an abelian group $\mathcal{G}$ is a normal subgroup, thus its normalizer $\mathcal{N}_{\mathcal{G}}(\mathcal{H})$ is $\mathcal{G}$ itself and $\mathcal{H}$ is only conjugate to itself.

(ii) The conjugacy classes of the symmetry group $3m$ of an equilateral triangle are $\{1\}$, $\{m_{10}, m_{01}, m_{11}\}$ and $\{3^+, 3^-\}$. The centralizer of m_{10} is just the group $\langle m_{10}\rangle$ generated by m_{10}, *i.e.* 1 and m_{10} are the only elements of $3m$ commuting with m_{10}. Analogously, one sees that the centralizer $\mathcal{C}_{\mathcal{G}}(h) = \langle h\rangle$ for each element h in $3m$, except for $h = 1$.

The subgroups $\langle m_{10}\rangle$, $\langle m_{01}\rangle$ and $\langle m_{11}\rangle$ are conjugate subgroups (with conjugating elements 3^+ and 3^-). These subgroups coincide with their normalizers, since they have index 3 in the full group.

(iii) The conjugacy classes of the symmetry group $4mm$ of a square are $\{1\}$, $\{2\}$, $\{m_{10}, m_{01}\}$, $\{m_{11}, m_{1\bar{1}}\}$ and $\{4^+, 4^-\}$. Since 2 forms a conjugacy class on its own, this is an element in the centre of $4mm$ and its centralizer is the full group. The centralizer of m_{10} is $\langle m_{10}, m_{01}\rangle$, which is also the centralizer of m_{01} (note that m_{10} and m_{01} are reflections with normal vectors perpendicular to each other, and thus commute). Analogously, $\langle m_{11}, m_{1\bar{1}}\rangle$ is the centralizer of both m_{11} and $m_{1\bar{1}}$. Finally, 4^+ only commutes with the rotations in $4mm$, therefore its centralizer is $\langle 4^+\rangle$.

The five subgroups of order 2 in $4mm$ fall into three conjugacy classes, namely the normal subgroup $\langle 2\rangle$ and the two pairs $\{\langle m_{10}\rangle, \langle m_{01}\rangle\}$ and $\{\langle m_{11}\rangle, \langle m_{1\bar{1}}\rangle\}$. The normalizer of both $\langle m_{10}\rangle$ and $\langle m_{01}\rangle$ is $\langle 2, m_{10}\rangle$ and the normalizer of both $\langle m_{11}\rangle$ and $\langle m_{1\bar{1}}\rangle$ is $\langle 2, m_{11}\rangle$.

In the context of crystallographic groups, conjugate subgroups are not only isomorphic, but have the same types of geometric elements, possibly with different directions. In many situations it is therefore sufficient to restrict attention to representatives of the conjugacy classes of subgroups. Furthermore, conjugation with elements from the normalizer of a group $\mathcal{H}$ permutes the geometric elements of the symmetry operations of $\mathcal{H}$. The role of the normalizer may in this situation be expressed by the phrase

The normalizer describes the *symmetry of the symmetries*.

Thus, the normalizer reflects an intrinsic ambiguity between different but equivalent descriptions of an object by its symmetries.

Example

The subgroup $\mathcal{H} = \langle 2, m_{10}\rangle$ is a normal subgroup of the symmetry group $\mathcal{G} = 4mm$ of the square, and thus $\mathcal{G}$ is the normalizer of $\mathcal{H}$ in $\mathcal{G}$. As can be seen in the diagram in Fig. 1.1.7.1, the fourfold rotation 4^+ maps the geometric element of the reflection m_{10} to the geometric element of m_{01} and *vice versa*, and fixes the geometric element of the rotation 4^+. Consequently, conjugation by 4^+ fixes $\mathcal{H}$ as a set, but interchanges the reflections m_{10} and m_{01}. These two reflections are geometrically indistinguishable, since their geometric elements are both lines through the centres of opposite edges of the square.

Analogously, 4^+ interchanges the geometric elements of the reflections m_{11} and $m_{1\bar{1}}$ of the subgroup $\mathcal{H}' = \langle 2, m_{11}\rangle$. These are the two reflection lines through opposite corners of the square.

In contrast to that, $\mathcal{G}$ does not contain an element mapping the geometric element of m_{10} to that of m_{11}. Note that an eightfold rotation would be such an element, but this is, however, not a symmetry of the square. The reflections m_{10} and m_{11} are thus geometrically different symmetry operations of the square.

References

Armstrong, M. A. (2010). *Groups and Symmetry*. New York: Springer.

Hill, V. E. (1999). *Groups and Characters.* Boca Raton: Chapman & Hall/CRC.

Müller, U. (2013). *Symmetry Relationships between Crystal Structures*. Oxford: IUCr/Oxford University Press.

Sternberg, S. (2008). *Group Theory and Physics*. Cambridge University Press.