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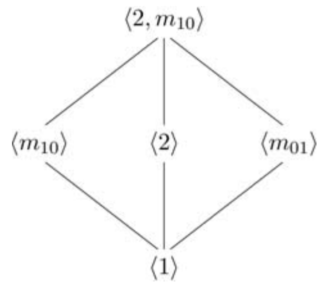

Figure 1.1.3.2
Subgroup diagram for the symmetry group 2*mm* of a rectangle.

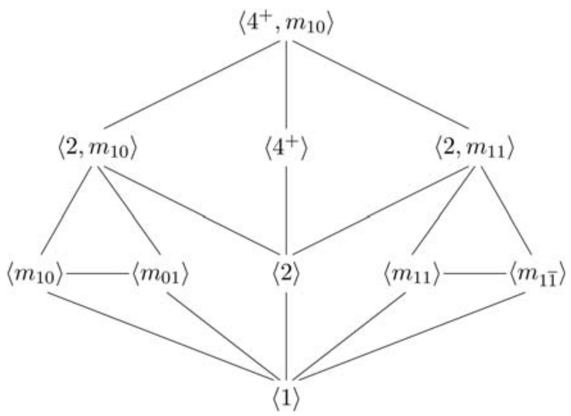

Figure 1.1.3.3
Subgroup diagram for the symmetry group 4*mm* of the square.

(vii) In the symmetry group 4*mm* of the square, the reflections m_{10} and m_{01} together with their product 2 and the identity element 1 form a subgroup of order 4. This subgroup can be recognized in the subgroup diagram of 4*mm* as the subdiagram of the subgroups of $\langle 2, m_{10}\rangle$ in the left part of Fig. 1.1.3.3 which coincides with the subgroup diagram of 2*mm* in Fig. 1.1.3.2. A different subgroup of order 4 is formed by the other pair of perpendicular reflections m_{11}, $m_{1\bar{1}}$ together with 2 and 1 and a third subgroup of order 4 is the cyclic subgroup $\langle 4^+\rangle$ generated by the fourfold rotation (see Fig. 1.1.3.3).

1.1.4. Cosets

A subgroup allows us to partition a group into disjoint subsets of the same size, called *cosets*.

Definition. Let $\mathcal{H} = \{h_1, h_2, h_3, \ldots\}$ be a subgroup of $\mathcal{G}$. Then for $g \in \mathcal{G}$ the set

$$g\mathcal{H} := \{gh_1, gh_2, gh_3, \ldots\} = \{gh \mid h \in \mathcal{H}\}$$

is called the *left coset* of $\mathcal{H}$ with *representative* g. Analogously, the *right coset* with representative g is defined as

$$\mathcal{H}g := \{h_1 g, h_2 g, h_3 g, \ldots\} = \{hg \mid h \in \mathcal{H}\}.$$

The coset $e\mathcal{H} = \mathcal{H} = \mathcal{H}e$ is called the *trivial coset* of $\mathcal{H}$.

Remarks

(i) Since two elements gh and gh' in the same coset $g\mathcal{H}$ can only be the same if $h = h'$, the elements of $g\mathcal{H}$ are in one-to-one correspondence with the elements of $\mathcal{H}$. In particular, for a finite subgroup $\mathcal{H}$ the number of elements in each coset of $\mathcal{H}$ equals the order $|\mathcal{H}|$ of the subgroup $\mathcal{H}$.

(ii) Every element contained in $g\mathcal{H}$ may serve as representative for this coset, *i.e.* $g'\mathcal{H} = g\mathcal{H}$ for every $g' \in g\mathcal{H}$. In particular, if an element g'' is contained in the intersection $g\mathcal{H} \cap g'\mathcal{H}$ of two cosets, one has $g''\mathcal{H} = g\mathcal{H}$ and $g''\mathcal{H} = g'\mathcal{H}$. This implies that two cosets are either disjoint (*i.e.* contain no common element) or they are equal.

These two remarks have an important consequence: since an element $g \in \mathcal{G}$ is contained in the coset $g\mathcal{H}$, the cosets of $\mathcal{H}$ partition the elements of $\mathcal{G}$ into sets of the same cardinality as $\mathcal{H}$ (which is of the order of $\mathcal{H}$ in the case where this is finite).

Definition. If the number of different cosets of a subgroup $\mathcal{H} \le \mathcal{G}$ is finite, this number is called the *index* of $\mathcal{H}$ in $\mathcal{G}$, denoted by $[i]$ or $[\mathcal{G} : \mathcal{H}]$. Otherwise, $\mathcal{H}$ is said to have *infinite index* in $\mathcal{G}$.

In the case of a finite group, the partitioning of the elements of $\mathcal{G}$ into the cosets of $\mathcal{H}$ shows that both the order of $\mathcal{H}$ and the index of $\mathcal{H}$ in $\mathcal{G}$ divide the order of $\mathcal{G}$. This is summarized in the following famous result.

Lagrange's theorem

For a finite group $\mathcal{G}$ and a subgroup $\mathcal{H}$ of $\mathcal{G}$ one has

$$|\mathcal{G}| = |\mathcal{H}| \cdot [\mathcal{G} : \mathcal{H}],$$

i.e. the order of a subgroup multiplied by its index gives the order of the full group.

For example, a group of order n cannot have a proper subgroup of order larger than $n/2$.

Whether or not two cosets of a subgroup $\mathcal{H}$ are equal depends on whether the quotient of their representatives is contained in $\mathcal{H}$: for left cosets one has $g\mathcal{H} = g'\mathcal{H}$ if and only if $g^{-1}g' \in \mathcal{H}$ and for right cosets $\mathcal{H}g = \mathcal{H}g'$ if and only if $g'g^{-1} \in \mathcal{H}$.

Definition. If $\mathcal{H}$ is a subgroup of $\mathcal{G}$ and $g_1, g_2, g_3, \ldots \in \mathcal{G}$ are such that $g_i\mathcal{H} \ne g_j\mathcal{H}$ for $i \ne j$, and every $g \in \mathcal{G}$ is contained in some left coset $g_i\mathcal{H}$, then $g_1, g_2, g_3, \ldots$ is called a system of *left coset representatives* of $\mathcal{G}$ relative to $\mathcal{H}$. It is customary to choose $g_1 = e$ so that the coset $g_1\mathcal{H} = e\mathcal{H} = \mathcal{H}$ is the subgroup $\mathcal{H}$ itself. The decomposition

$$\mathcal{G} = \mathcal{H} \cup g_2\mathcal{H} \cup g_3\mathcal{H} \ldots$$

is called the *coset decomposition* of $\mathcal{G}$ into left cosets relative to $\mathcal{H}$.

Analogously, $g_1{}', g_2{}', g_3{}', \ldots \in \mathcal{G}$ is called a system of *right coset representatives* if $\mathcal{H}g_i{}' \ne \mathcal{H}g_j{}'$ for $i \ne j$ and every $g \in \mathcal{G}$ is contained in some right coset $\mathcal{H}g_i$. Again, one usually chooses $g_1{}' = e$ and the decomposition

$$\mathcal{G} = \mathcal{H} \cup \mathcal{H}g_2{}' \cup \mathcal{H}g_3{}' \ldots$$

is called the *coset decomposition* of $\mathcal{G}$ into right cosets relative to $\mathcal{H}$.

To obtain the coset decomposition one starts by choosing $\mathcal{H}$ as the first coset (with representative e). Next, an element $g_2 \in \mathcal{G}$ with $g_2 \notin \mathcal{H}$ is selected as representative for the second coset $g_2\mathcal{H}$. For the third coset, an element $g_3 \in \mathcal{G}$ with $g_3 \notin \mathcal{H}$ and $g_3 \notin g_2\mathcal{H}$ is required. If at a certain stage the cosets $\mathcal{H}, g_2\mathcal{H}, \ldots, g_m\mathcal{H}$ have been defined but do not yet exhaust $\mathcal{G}$, an element g_{m+1} not contained in the union $\mathcal{H} \cup g_2\mathcal{H} \cup \ldots \cup g_m\mathcal{H}$ is chosen as representative for the next coset.

Examples

(i) Let $\mathcal{G} = 3m$ be the symmetry group of an equilateral triangle and $\mathcal{H} = \langle 3^+\rangle$ its subgroup containing the rotations. Then for every reflection $m \in \mathcal{G}$ the elements e, m form a system of coset representatives of $\mathcal{G}$ relative to $\mathcal{H}$ and the coset decomposition is $\mathcal{G} = \{1, 3^+, 3^-\} \cup \{m_{10}, m_{01}, m_{11}\}$.

(ii) For any integer n, the set $n\mathbb{Z} := \{na \mid a \in \mathbb{Z}\}$ of multiples of n forms an infinite subgroup of index n in $\mathbb{Z}$. A system of coset representatives of $\mathbb{Z}$ relative to $n\mathbb{Z}$ is formed by the numbers $0, 1, 2, \ldots, n-1$. The coset with representative 0 is $\{\ldots, -n, 0, n, 2n, \ldots\}$, the coset with representative 1 is $\{\ldots, -n+1, 1, n+1, 2n+1, \ldots\}$ and an integer a belongs to the coset with representative k if and only if a gives remainder k upon division by n.

1.1.5. Normal subgroups, factor groups

In general, the left and right cosets of a subgroup $\mathcal{H}$ differ, for example in the symmetry group $3m$ of an equilateral triangle the left coset decomposition with respect to the subgroup $\mathcal{H} = \{1, m_{10}\}$ is

$$\begin{aligned}&\{1, m_{10}\} \cup 3^+\{1, m_{10}\} \cup 3^-\{1, m_{10}\}\\ =&\{1, m_{10}\} \cup \{3^+, m_{11}\} \cup \{3^-, m_{01}\},\end{aligned}$$

whereas the right coset decomposition is

$$\begin{aligned}&\{1, m_{10}\} \cup \{1, m_{10}\}3^+ \cup \{1, m_{10}\}3^-\\ =&\{1, m_{10}\} \cup \{3^+, m_{01}\} \cup \{3^-, m_{11}\}.\end{aligned}$$

For particular subgroups, however, it turns out that the left and right cosets coincide, *i.e.* one has $g\mathcal{H} = \mathcal{H}g$ for all $g \in \mathcal{G}$. This means that for every $h \in \mathcal{H}$ and every $g \in \mathcal{G}$ the element gh is of the form $gh = h'g$ for some $h' \in \mathcal{H}$ and thus $ghg^{-1} = h' \in \mathcal{H}$. The element $h' = ghg^{-1}$ is called the *conjugate of h by g*. Note that in the definition of the conjugate element there is a choice whether the inverse element g^{-1} is placed to the left or right of h. Depending on the applications that are envisaged and on the preferences of the author, both versions ghg^{-1} and $g^{-1}hg$ are found in the literature, but in the context of crystallographic groups it is more convenient to have the inverse g^{-1} to the right of h.

An important aspect of conjugate elements is that they share many properties, such as the order or the type of symmetry operation. As a consequence, conjugate symmetry operations have the same type of geometric elements. For example, if h is a threefold rotation in three-dimensional space, its geometric element is the line along the rotation axis. The geometric element of a conjugate element ghg^{-1} is then also a line fixed by a threefold rotation, but in general this line has a different direction.

Definition. A subgroup $\mathcal{H}$ of $\mathcal{G}$ is called a *normal subgroup* if $ghg^{-1} \in \mathcal{H}$ for all $g \in \mathcal{G}$ and all $h \in \mathcal{H}$. This is denoted by $\mathcal{H} \trianglelefteq \mathcal{G}$. For a normal subgroup $\mathcal{H}$, the left and right cosets of $\mathcal{G}$ with respect to $\mathcal{H}$ coincide.

Remarks

(i) The full group $\mathcal{G}$ and the trivial subgroup $\{e\}$ are always normal subgroups of $\mathcal{G}$. These are often called the *trivial* normal subgroups of $\mathcal{G}$.

(ii) In abelian groups, every subgroup is a normal subgroup, because $gh = hg$ implies $ghg^{-1} = h \in \mathcal{H}$.

(iii) A subgroup $\mathcal{H}$ of index 2 in $\mathcal{G}$ is always a normal subgroup, since the coset decomposition relative to $\mathcal{H}$ consists of only two cosets and for any element $g \notin \mathcal{H}$ the left and right cosets $g\mathcal{H}$ and $\mathcal{H}g$ both consist precisely of those elements of $\mathcal{G}$ that are not contained in $\mathcal{H}$. Therefore, $g\mathcal{H} = \mathcal{H}g$ for $g \notin \mathcal{H}$ and for $h \in \mathcal{H}$ clearly $h\mathcal{H} = \mathcal{H} = \mathcal{H}h$ holds.

(iv) In order to check whether a subgroup $\mathcal{H}$ of $\mathcal{G}$ is a normal subgroup it is sufficient to check whether $ghg^{-1} \in \mathcal{H}$ for generators g of $\mathcal{G}$ and generators h of $\mathcal{H}$. This is due to the fact that on the one hand $(g_1g_2)\,h\,(g_1g_2)^{-1} = g_1(g_2hg_2^{-1})g_1^{-1}$ and on the other hand $g(h_1h_2)g^{-1} = (gh_1g^{-1})(gh_2g^{-1})$.

Examples

(i) In the symmetry group $3m$ of an equilateral triangle, the subgroup generated by the threefold rotation 3^+ is a normal subgroup because it is of index 2 in $3m$.
The subgroups of order 2 generated by the reflections m_{10}, m_{01} and m_{11} are not normal because $3^+ \cdot m_{10} \cdot 3^- = m_{01} \notin \langle m_{10}\rangle$, $3^+ \cdot m_{01} \cdot 3^- = m_{11} \notin \langle m_{01}\rangle$ and $3^+ \cdot m_{11} \cdot 3^- = m_{10} \notin \langle m_{11}\rangle$.

(ii) In the symmetry group $4mm$ of the square, the subgroups $\langle 2, m_{10}\rangle$, $\langle 4^+\rangle$, and $\langle 2, m_{11}\rangle$ are normal subgroups because they are subgroups of index 2.
The subgroups of order 2 generated by the reflections m_{10}, m_{01}, m_{11} and $m_{1\bar{1}}$ are not normal because $4^+ \cdot m_{10} \cdot 4^- = m_{01} \notin \langle m_{10}\rangle$, $4^+ \cdot m_{01} \cdot 4^- = m_{10} \notin \langle m_{01}\rangle$, $4^+ \cdot m_{11} \cdot 4^- = m_{1\bar{1}} \notin \langle m_{11}\rangle$ and $4^+ \cdot m_{1\bar{1}} \cdot 4^- = m_{11} \notin \langle m_{1\bar{1}}\rangle$.
The subgroup of order 2 generated by the twofold rotation 2 is normal because $4^+ \cdot 2 \cdot 4^- = 2$ and $m_{10} \cdot 2 \cdot m_{10}^{-1} = 2$.

For a subgroup $\mathcal{H}$ of $\mathcal{G}$ and an element $g \in \mathcal{G}$, the conjugates ghg^{-1} form a subgroup

$$\mathcal{H}' = g\mathcal{H}g^{-1} = \{ghg^{-1} \mid h \in \mathcal{H}\}$$

because $gh_1g^{-1} \cdot gh_2g^{-1} = gh_1h_2g^{-1}$. This subgroup is called the *conjugate subgroup* of $\mathcal{H}$ by g. As already noted, conjugation does not alter the type of symmetry operations and their geometric elements, but it is possible that the orientations of the geometric elements are changed.

Using the concept of conjugate subgroups, a normal subgroup is a subgroup $\mathcal{H}$ that coincides with all its conjugate subgroups $g\mathcal{H}g^{-1}$. This means that the set of geometric elements of a normal subgroup is not changed by conjugation; the single geometric elements may, however, be permuted by the conjugating element. In the example of the symmetry group $4mm$ discussed above, the normal subgroup $\langle 2, m_{10}\rangle$ contains the reflections m_{10} and m_{01} with the lines along the coordinate axes as geometric elements. These two lines are interchanged by the fourfold rotation 4^+, corresponding to the fact that conjugation by 4^+ interchanges m_{10} and m_{01}. The concept of conjugation will be discussed in more detail in Section 1.1.8.

One of the main motivations for studying normal subgroups is that they allow us to define a group operation on the cosets of $\mathcal{H}$ in $\mathcal{G}$. The products of any element in the coset $g\mathcal{H}$ with any element in the coset $g'\mathcal{H}$ lie in a single coset, namely in the coset $gg'\mathcal{H}$. Thus we can define the product of the two cosets $g\mathcal{H}$ and $g'\mathcal{H}$ as the coset with representative gg'.

Definition. The set $\mathcal{G}/\mathcal{H} := \{g\mathcal{H} \mid g \in \mathcal{G}\}$ together with the binary operation

$$g\mathcal{H} \circ g'\mathcal{H} := gg'\mathcal{H}$$

forms a group, called the *factor group* or *quotient group* of $\mathcal{G}$ by $\mathcal{H}$.

The identity element of the factor group $\mathcal{G}/\mathcal{H}$ is the coset $\mathcal{H}$ and the inverse element of $g\mathcal{H}$ is the coset $g^{-1}\mathcal{H}$.

A familiar example of a factor group is provided by the times on a clock. If it is 8 o'clock (in the morning) now, then we say that in nine hours it will be 5 o'clock (in the afternoon). We regard times as elements of the factor group $\mathbb{Z}/12\mathbb{Z}$ in which

references