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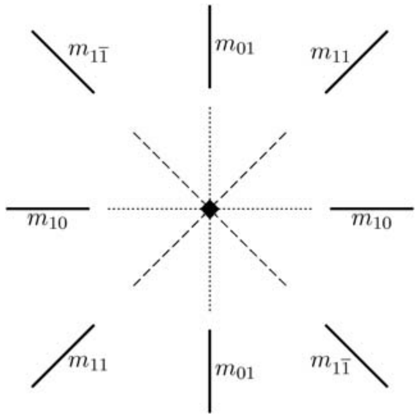

Figure 1.1.5.1
Symmetry group of an eightfold star.

$(8 + 12\mathbb{Z}) + (9 + 12\mathbb{Z}) = 17 + 12\mathbb{Z} = 5 + 12\mathbb{Z}$. In the factor group $\mathbb{Z}/12\mathbb{Z}$, the clock is imagined as a circle of circumference 12 around which the line of integers is wrapped so that integers with a difference of 12 are located at the same position on the circle.

The clock example is a special case of factor groups of the integers. We have already seen that the set $n\mathbb{Z} = \{na \mid a \in \mathbb{Z}\}$ of multiples of a natural number n forms a subgroup of index n in $\mathbb{Z}$. This is a normal subgroup, since $\mathbb{Z}$ is an abelian group. The factor group $\mathbb{Z}/n\mathbb{Z}$ represents the addition of integers *modulo n*.

Examples

(i) If we take $\mathcal{G}$ to be the symmetry group $4mm$ of the square and choose as normal subgroup the subgroup $\mathcal{H} = \langle 4^+ \rangle$ generated by the fourfold rotation, we obtain a factor group $\mathcal{G}/\mathcal{H}$ with two elements, namely the cosets $\mathcal{H} = \{1, 2, 4^+, 4^-\}$ and $m_{10}\mathcal{H} = \{m_{10}, m_{01}, m_{11}, m_{1\bar{1}}\}$. The trivial coset $\mathcal{H}$ is the identity element in the factor group $\mathcal{G}/\mathcal{H}$ and contains the rotations in $4mm$. The other element $m_{10}\mathcal{H}$ in the factor group $\mathcal{G}/\mathcal{H}$ consists of the reflections in $4mm$.
In this example, the separation of the rotations and reflections in $4mm$ into the two cosets $\mathcal{H}$ and $m_{10}\mathcal{H}$ makes it easy to see that the product of two cosets is independent of the chosen representative of the coset: the product of two rotations is again a rotation, hence $\mathcal{H} \cdot \mathcal{H} = \mathcal{H}$, the product of a rotation and a reflection is a reflection, hence $\mathcal{H} \cdot m_{10}\mathcal{H} = m_{10}\mathcal{H} \cdot \mathcal{H} = m_{10}\mathcal{H}$, and finally the product of two reflections is a rotation, hence $m_{10}\mathcal{H} \cdot m_{10}\mathcal{H} = \mathcal{H}$. The multiplication table of the factor group is thus

	$\mathcal{H}$	$m_{10}\mathcal{H}$
$\mathcal{H}$	$\mathcal{H}$	$m_{10}\mathcal{H}$
$m_{10}\mathcal{H}$	$m_{10}\mathcal{H}$	$\mathcal{H}$

(ii) The symmetry group of a square is the same as the symmetry group of an eightfold star, as shown in Fig. 1.1.5.1. If we regard the star as being built from four lines (two dotted and two dashed), then the twofold rotation does not move any of the lines, it only interchanges the points within each line (symmetric with respect to the centre). Regarding the lines as sets of points, the twofold rotation thus does not change anything. The effects of the different symmetry operations on the lines of the eightfold star are then precisely given by the factor group $\mathcal{G}/\mathcal{H}$, where $\mathcal{G}$ is the symmetry group $4mm$ of the square and $\mathcal{H}$ is the normal subgroup generated by the twofold rotation 2: the cosets relative to $\mathcal{H}$ are $\{1, 2\}$, $\{4^+, 4^-\}$, $\{m_{10}, m_{01}\}$, $\{m_{11}, m_{1\bar{1}}\}$, and these cosets collect together the elements of $4mm$ that have the same effect on the lines of the eightfold star. For example, both 4^+ and 4^- interchange both the two dotted and the two dashed lines, m_{10} and m_{01} both interchange the two dashed lines but fix the two dotted lines and m_{11} and $m_{1\bar{1}}$ both interchange the two dotted lines but fix the two dashed lines. Owing to the fact that $\mathcal{H}$ is a normal subgroup, the product of elements from two cosets always lies in the same coset, independent of which elements are chosen from the two cosets. For example, the product of an element from the coset $\{4^+, 4^-\}$ with an element of the coset $\{m_{10}, m_{01}\}$ always gives an element of the coset $\{m_{11}, m_{1\bar{1}}\}$. Working out the products for all pairs of cosets, one obtains the following multiplication table for the factor group $\mathcal{G}/\mathcal{H}$:

	$\{1, 2\}$	$\{4^+, 4^-\}$	$\{m_{10}, m_{01}\}$	$\{m_{11}, m_{1\bar{1}}\}$
$\{1, 2\}$	$\{1, 2\}$	$\{4^+, 4^-\}$	$\{m_{10}, m_{01}\}$	$\{m_{11}, m_{1\bar{1}}\}$
$\{4^+, 4^-\}$	$\{4^+, 4^-\}$	$\{1, 2\}$	$\{m_{11}, m_{1\bar{1}}\}$	$\{m_{10}, m_{01}\}$
$\{m_{10}, m_{01}\}$	$\{m_{10}, m_{01}\}$	$\{m_{11}, m_{1\bar{1}}\}$	$\{1, 2\}$	$\{4^+, 4^-\}$
$\{m_{11}, m_{1\bar{1}}\}$	$\{m_{11}, m_{1\bar{1}}\}$	$\{m_{10}, m_{01}\}$	$\{4^+, 4^-\}$	$\{1, 2\}$

(iii) If one takes cosets with respect to a subgroup that is not normal, the products of elements from two cosets do not lie in a single coset. As we have seen, the left cosets of the group $3m$ of an equilateral triangle with respect to the non-normal subgroup $\mathcal{H} = \{1, m_{10}\}$ are $\{1, m_{10}\}$, $\{3^+, m_{11}\}$ and $\{3^-, m_{01}\}$. Taking products from elements of the first and second coset, we get $1 \cdot 3^+ = 3^+$ and $1 \cdot m_{11} = m_{11}$, which are both in the second coset, but $m_{10} \cdot 3^+ = m_{01}$ and $m_{10} \cdot m_{11} = 3^-$, which are both in the third coset.

1.1.6. Homomorphisms, isomorphisms

In order to relate two groups, mappings between the groups that are compatible with the group operations are very useful.

Recall that a *mapping* φ from a set A to a set B associates to each $a \in A$ an element $b \in B$, denoted by $\varphi(a)$ and called the *image* of a (under φ).

Definition. For two groups $\mathcal{G}$ and $\mathcal{H}$, a mapping φ from $\mathcal{G}$ to $\mathcal{H}$ is called a *group homomorphism* or *homomorphism* for short, if it is compatible with the group operations in $\mathcal{G}$ and $\mathcal{H}$, *i.e.* if

$$\varphi(gg') = \varphi(g)\varphi(g') \text{ for all } g, g' \text{ in } \mathcal{G}.$$

The compatibility with the group operation is captured in the phrase

The image of the product is equal to the product of the images.

Fig. 1.1.6.1 gives a schematic description of the definition of a homomorphism. For φ to be a homomorphism, the two curved arrows are required to give the same result, *i.e.* first multiplying two elements in $\mathcal{G}$ and then mapping the product to $\mathcal{H}$ must be the same as first mapping the elements to $\mathcal{H}$ and then multiplying them.

It follows from the definition of a homomorphism that the identity element of $\mathcal{G}$ must be mapped to the identity element of $\mathcal{H}$ and that the inverse g^{-1} of an element $g \in \mathcal{G}$ must be mapped to the inverse of the image of g, *i.e.* that $\varphi(g^{-1}) = \varphi(g)^{-1}$. In general, however, other elements than the identity element may also be mapped to the identity element of $\mathcal{H}$.

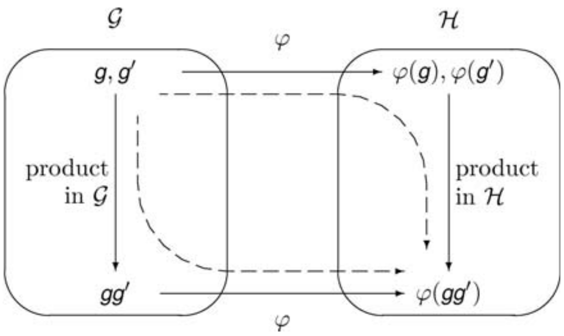

Figure 1.1.6.1
Schematic description of a homomorphism.

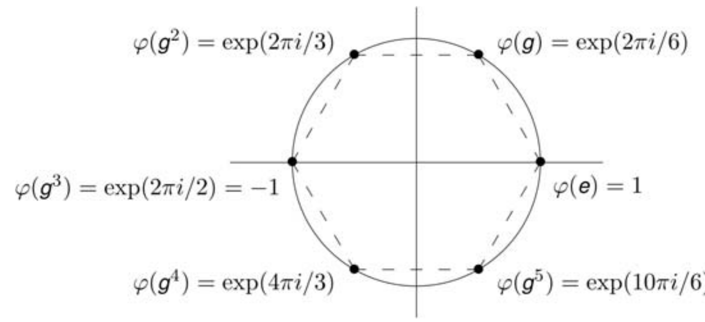

Figure 1.1.6.2
Cyclic group of order 6 embedded in the group of the unit circle.

Definition. Let φ be a group homomorphism from $\mathcal{G}$ to $\mathcal{H}$.

(i) The set $\{g \in \mathcal{G} \mid \varphi(g) = e\}$ of elements mapped to the identity element of $\mathcal{H}$ is called the *kernel* of φ, denoted by $\ker\varphi$.

(ii) The set $\varphi(\mathcal{G}) := \{\varphi(g) \mid g \in \mathcal{G}\}$ is called the *image of* $\mathcal{G}$ *under* φ.

In the case where only the identity element of $\mathcal{G}$ lies in the kernel of φ, one can conclude that $\varphi(g) = \varphi(g')$ implies $g = g'$ and φ is called an *injective homomorphism*. In this situation no information about the group $\mathcal{G}$ is lost and the homomorphism φ can be regarded as an *embedding* of $\mathcal{G}$ into $\mathcal{H}$.

The image $\varphi(\mathcal{G})$ of any homomorphism from $\mathcal{G}$ to $\mathcal{H}$ forms not just a subset, but a subgroup of $\mathcal{H}$. It is not required that $\varphi(\mathcal{G})$ is all of $\mathcal{H}$, but if this happens to be the case, φ is called a *surjective homomorphism*.

Examples

(i) For the symmetry group $4mm$ of the square a homomorphism φ to a cyclic group $\mathcal{C}_2 = \{e, g\}$ of two elements is given by $\varphi(1) = \varphi(4^+) = \varphi(2) = \varphi(4^-) = e$ and $\varphi(m_{10}) = \varphi(m_{01}) = \varphi(m_{11}) = \varphi(m_{1\bar{1}}) = g$, *i.e.* by mapping the rotations in $4mm$ to the identity element of $\mathcal{C}_2$ and the reflections to the non-trivial element.
Since every element of $\mathcal{C}_2$ is the image of some element of $4mm$, φ is a surjective homomorphism, but it is not injective because the kernel consists of all rotations in $4mm$ and not only of the identity element.

(ii) The cyclic group $\mathcal{C}_n = \{e, g, g^2, \ldots, g^{n-1}\}$ of order n is mapped into the (multiplicative) group S^1 of the unit circle in the complex plane by mapping g^k to $\exp(2\pi ik/n)$. As displayed in Fig. 1.1.6.2, the image of $\mathcal{C}_n$ under this homomorphism are points on the unit circle which form the corners of a regular n-gon.
This is an injective homomorphism because the smallest $k > 0$ with $\exp(2\pi ik/n) = 1$ is $k = n$ and $g^n = e$ in $\mathcal{C}_n$, thus by this homomorphism $\mathcal{C}_n$ can be regarded as a subgroup of S^1. It is clear that φ cannot be surjective, because S^1 is an infinite group and the image $\varphi(\mathcal{C}_n)$ consists of only finitely many elements.

(iii) For the additive group $(\mathbb{Z}, +)$ of integers and a cyclic group $\mathcal{C}_n = \{e, g, g^2, \ldots, g^{n-1}\}$, for every integer q a homomorphism φ is defined by mapping $1 \in \mathbb{Z}$ to g^q, which gives $\varphi(a) = g^{qa}$ for $a \in \mathbb{Z}$. This is never an injective homomorphism, because $n\mathbb{Z} = \{na \mid a \in \mathbb{Z}\}$ is contained in the kernel of φ. Whether or not φ is surjective depends on whether g^q is a generator of $\mathcal{C}_n$. This is the case if and only if n and q have no non-trivial common divisors.

Definition. A homomorphism φ from $\mathcal{G}$ to $\mathcal{H}$ is called an *isomorphism* if $\ker\varphi = \{e\}$ and $\varphi(\mathcal{G}) = \mathcal{H}$, *i.e.* if φ is both injective and surjective. An isomorphism is thus a one-to-one mapping between the elements of $\mathcal{G}$ and $\mathcal{H}$ which is also a homomorphism.

Groups $\mathcal{G}$ and $\mathcal{H}$ between which an isomorphism exist are called *isomorphic groups*, this is denoted by $\mathcal{G} \cong \mathcal{H}$.

Isomorphic groups may differ in the way they are realized, but they coincide in their structure. In essence, one can regard isomorphic groups as the same group with different names or labels for the group elements. For example, isomorphic groups have the same multiplication table if the elements are relabelled according to the isomorphism identifying the elements of the first group with those of the second. If one wants to stress that a certain property of a group $\mathcal{G}$ will be the same for all groups which are isomorphic to $\mathcal{G}$, one speaks of $\mathcal{G}$ as an *abstract group*.

Examples

(i) The symmetry group $3m$ of an equilateral triangle is isomorphic to the group S_3 of all permutations of $\{1, 2, 3\}$. This can be seen as follows: labelling the corners of the triangle by $1, 2, 3$, each element of $3m$ gives rise to a permutation of the labels and mapping an element to the corresponding permutation is a homomorphism. The only element fixing all three corners of the triangle is the identity element of $3m$, thus the homomorphism is injective. On the other hand, the groups $3m$ and S_3 both have 6 elements, hence the homomorphism is also surjective, and thus it is an isomorphism.

(ii) For the symmetry group $\mathcal{G} = 4mm$ of the square and its normal subgroup $\mathcal{H}$ generated by the fourfold rotation, the factor group $\mathcal{G}/\mathcal{H}$ is isomorphic to a cyclic group $\mathcal{C}_2 = \{e, g\}$ of order 2. The trivial coset (containing the rotations in $4mm$) corresponds to the identity element e, the other coset (containing the reflections) corresponds to g.

(iii) The real numbers $\mathbb{R}$ form a group with addition as operation and the positive real numbers $\mathbb{R}_{>0} = \{x \in \mathbb{R} \mid x > 0\}$ form a group with multiplication as operation. The exponential mapping $x \mapsto \exp(x)$ is a homomorphism from $(\mathbb{R}, +)$ to $(\mathbb{R}_{>0}, \cdot)$ because $\exp(x + y) = \exp(x) \cdot \exp(y)$. It is an injective homomorphisms because $\exp(x) = 1$ only for $x = 0$ [which is the identity element in $(\mathbb{R}, +)$] and it is a surjective homomorphism because for any $y > 0$ there is an $x \in \mathbb{R}$ with $\exp(x) = y$, namely $x = \log(y)$. The exponential mapping therefore provides an isomorphism from $(\mathbb{R}, +)$ to $(\mathbb{R}_{>0}, \cdot)$.

The kernel of a homomorphism φ is always a normal subgroup, since for $h \in \ker\varphi$ and $g \in \mathcal{G}$ one has $\varphi(ghg^{-1}) = \varphi(g)\varphi(h)\varphi(g^{-1}) = \varphi(g)\varphi(g^{-1}) = e$. The information about the elements in the kernel of φ is lost after applying φ, because they are all mapped to the identity element of $\mathcal{H}$. More precisely, if $\mathcal{N} = \ker\varphi$, then all elements from the coset $g\mathcal{N}$ are mapped to the same element $\varphi(g)$ in $\mathcal{H}$, since for $n \in \mathcal{N}$ one has $\varphi(gn) = \varphi(g)\varphi(n) = \varphi(g)$. Conversely, if elements are mapped to the same element, they have to lie in the same coset, since $\varphi(g) = \varphi(g')$ implies $\varphi(g^{-1}g') = e$, thus $g^{-1}g' \in \mathcal{N}$ and thus $g^{-1}g'\mathcal{N} = \mathcal{N}$, *i.e.* $g\mathcal{N} = g'\mathcal{N}$. The cosets of $\mathcal{N}$ therefore partition the elements of $\mathcal{G}$ according to their images under φ. This observation is summarized in the following result, which is one of the most powerful theorems in group theory.

Homomorphism theorem

Let φ be a homomorphism from $\mathcal{G}$ to $\mathcal{H}$ with kernel $\ker\varphi = \mathcal{N} \trianglelefteq \mathcal{G}$. Then the factor group $\mathcal{G}/\mathcal{N}$ is isomorphic to the image $\varphi(\mathcal{G})$ *via* the isomorphism $g\mathcal{N} \longmapsto \varphi(g)$.

Examples

(i) The homomorphism φ from $4mm$ to $\mathcal{C}_2 = \{e, g\}$ sending the rotations in $4mm$ to $e \in \mathcal{C}_2$ and the reflections to $g \in \mathcal{C}_2$ has the group $\mathcal{N} = \langle 4 \rangle$ of rotations in $4mm$ as its kernel. The factor group $4mm/\mathcal{N}$ has the cosets $\mathcal{N} = \{1, 4^+, 2, 4^-\}$ and $m_{10}\mathcal{N} = \{m_{10}, m_{01}, m_{11}, m_{1\bar{1}}\}$ as its elements and the homomorphism theorem confirms that mapping $\mathcal{N}$ to $e \in \mathcal{C}_2$ and $m_{10}\mathcal{N}$ to $g \in \mathcal{C}_2$ is an isomorphism from $4mm/\mathcal{N}$ to $\mathcal{C}_2$.

(ii) The homomorphism φ from the additive group $(\mathbb{Z}, +)$ of integers to the cyclic group $\mathcal{C}_n = \langle g \rangle$ mapping k to g^k has $\mathcal{N} = n\mathbb{Z} = \{na \mid a \in \mathbb{Z}\}$ as its kernel. Since φ is a surjective homomorphism, the homomorphism theorem states that the factor group $\mathbb{Z}/n\mathbb{Z}$ is isomorphic to the cyclic group $\mathcal{C}_n$. The operation in the factor group $\mathbb{Z}/n\mathbb{Z}$ is 'addition *modulo n*'.

1.1.7. Group actions

The concept of a group is the essence of an abstraction process which distils the common features of various examples of groups. On the other hand, although abstract groups are important and interesting objects in their own right, they are particularly useful because the group elements *act* on something, *i.e.* they can be applied to certain objects. For example, symmetry groups act on the points in space, but they also act on lines or planes. Groups of permutations act on the symbols themselves, but also on ordered and unordered pairs. Groups of matrices act on the vectors of a vector space, but also on the subspaces. All these different actions can be described in a uniform manner and common concepts can be developed.

Definition. A *group action* of a group $\mathcal{G}$ on a set $\Omega = \{\omega \mid \omega \in \Omega\}$ assigns to each pair (g, ω) an object $\omega' = g(\omega)$ of Ω such that the following hold:

(i) applying two group elements g and g' consecutively has the same effect as applying the product $g'g$, *i.e.* $g'(g(\omega)) = (g'g)(\omega)$ (note that since the group elements act *from the left* on the objects in Ω, the elements in a product of two (or more) group elements are applied right-to-left);

(ii) applying the identity element e of $\mathcal{G}$ has no effect on ω, *i.e.* $e(\omega) = \omega$ for all ω in Ω.

One says that the object ω is *moved* to $g(\omega)$ by g.

Example

The abstract group $\mathcal{C}_2 = \{e, g\}$ occurs as symmetry group in three-dimensional space with three different actions of g:

(i) If g is a reflection, then the points fixed by g form a two-dimensional plane.

(ii) If g is a twofold rotation, then the fixed points of g form a one-dimensional line.

(iii) If g is an inversion, then only a single point is fixed by g.

Often, two objects ω and ω' are regarded as equivalent if there is a group element moving ω to ω'. This notion of equivalence is in fact an *equivalence relation* in the strict mathematical sense:

(*a*) it is *reflexive*, *i.e.* ω is equivalent to itself: this is easily seen since $e(\omega) = \omega$;

(*b*) it is *symmetric*, *i.e.* if ω is equivalent to ω', then ω' is also equivalent to ω: this holds since $g(\omega) = \omega'$ implies $g^{-1}(\omega') = \omega$;

(*c*) it is *transitive*, *i.e.* if ω is equivalent to ω' and ω' is equivalent to ω'', then ω is equivalent to ω'': this is true because $g(\omega) = \omega'$ and $g'(\omega') = \omega''$ implies $g'g(\omega) = \omega''$.

Via this equivalence relation, the action of $\mathcal{G}$ partitions the objects in Ω into equivalence classes, where the equivalence class of an object $\omega \in \Omega$ consists of all objects which are equivalent to ω.

Definition. Two objects $\omega, \omega' \in \Omega$ lie in the same *orbit* under $\mathcal{G}$ if there exists $g \in \mathcal{G}$ such that $\omega' = g(\omega)$.

The set $\mathcal{G}(\omega) := \{g(\omega) \mid g \in \mathcal{G}\}$ of all objects in the orbit of ω is called the *orbit of ω under $\mathcal{G}$*.

The set $S_{\mathcal{G}}(\omega) := \{g \in \mathcal{G} \mid g(\omega) = \omega\}$ of group elements that do not move the object ω is a subgroup of $\mathcal{G}$ called the *stabilizer* of ω in $\mathcal{G}$.

If the orbit of a group action is finite, the length of the orbit is equal to the index of the stabilizer and thus in particular a divisor of the group order (in the case of a finite group). Actually, the objects in an orbit are in a very explicit one-to-one correspondence with the cosets relative to the stabilizer, as is summarized in the *orbit–stabilizer theorem*.

Orbit–stabilizer theorem

For a group $\mathcal{G}$ acting on a set Ω let ω be an object in Ω and let $S_{\mathcal{G}}(\omega)$ be the stabilizer of ω in $\mathcal{G}$.

(i) If $g_1 S_{\mathcal{G}}(\omega) \cup g_2 S_{\mathcal{G}}(\omega) \cup \ldots \cup g_m S_{\mathcal{G}}(\omega)$ is the coset decomposition of $\mathcal{G}$ relative to $S_{\mathcal{G}}(\omega)$, then the coset $g_i S_{\mathcal{G}}(\omega)$ consists of precisely those elements of $\mathcal{G}$ that move ω to $g_i(\omega)$. As a consequence, the full orbit of ω is already obtained by applying only the coset representatives to ω, *i.e.* $\mathcal{G}(\omega) = \{g_1(\omega), g_2(\omega), \ldots, g_m(\omega)\}$ and the number of cosets equals the length of the orbit.

(ii) For objects in the same orbit under $\mathcal{G}$, the stabilizers are *conjugate subgroups* of $\mathcal{G}$ (*cf.* Section 1.1.5). If $\omega' = g(\omega)$, then $S_{\mathcal{G}}(\omega') = g S_{\mathcal{G}}(\omega) g^{-1}$, *i.e.* the stabilizer of ω' is obtained by conjugating the stabilizer of ω by the element g moving ω to ω'.

Example

The symmetry group $\mathcal{G} = 4mm$ of the square acts on the corners of a square as displayed in Fig. 1.1.7.1. All four points lie in a single orbit under $\mathcal{G}$ and the stabilizer of the point 1 is $\mathcal{H} = \langle m_{1\bar{1}} \rangle$, *i.e.* a subgroup of index 4, as required by the orbit–stabilizer theorem. The stabilizers of the other points are conjugate to $\mathcal{H}$: The stabilizer of corner 3 equals $\mathcal{H}$ and the stabilizer of both the corners 2 and 4 is $\langle m_{11} \rangle$, which is conjugate to $\mathcal{H}$ by the fourfold rotation 4^+ which moves corner 1 to corner 2.

An n-dimensional space group $\mathcal{G}$ acts on the points of the n-dimensional space $\mathbb{R}^n$. The stabilizer of a point $P \in \mathbb{R}^n$ is called

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