The integration of a general function over $\mathbb{R}^m \times \mathbb{R}^n$ may be accomplished in two steps according to *Fubini's theorem*. Given $F \in L^1(\mathbb{R}^m \times \mathbb{R}^n)$, the functions

$$F_1 : \mathbf{x} \longmapsto \int_{\mathbb{R}^n} F(\mathbf{x}, \mathbf{y}) \, \mathrm{d}^n\mathbf{y}$$
$$F_2 : \mathbf{y} \longmapsto \int_{\mathbb{R}^m} F(\mathbf{x}, \mathbf{y}) \, \mathrm{d}^m\mathbf{x}$$

exist for almost all $\mathbf{x} \in \mathbb{R}^m$ and almost all $\mathbf{y} \in \mathbb{R}^n$, respectively, are integrable, and

$$\int_{\mathbb{R}^m \times \mathbb{R}^n} F(\mathbf{x}, \mathbf{y}) \, \mathrm{d}^m\mathbf{x} \, \mathrm{d}^n\mathbf{y} = \int_{\mathbb{R}^m} F_1(\mathbf{x}) \, \mathrm{d}^m\mathbf{x} = \int_{\mathbb{R}^n} F_2(\mathbf{y}) \, \mathrm{d}^n\mathbf{y}.$$

Conversely, if any one of the integrals

(i) $$\int_{\mathbb{R}^m \times \mathbb{R}^n} |F(\mathbf{x}, \mathbf{y})| \, \mathrm{d}^m\mathbf{x} \, \mathrm{d}^n\mathbf{y}$$

(ii) $$\int_{\mathbb{R}^m} \left(\int_{\mathbb{R}^n} |F(\mathbf{x}, \mathbf{y})| \, \mathrm{d}^n\mathbf{y} \right) \mathrm{d}^m\mathbf{x}$$

(iii) $$\int_{\mathbb{R}^n} \left(\int_{\mathbb{R}^m} |F(\mathbf{x}, \mathbf{y})| \, \mathrm{d}^m\mathbf{x} \right) \mathrm{d}^n\mathbf{y}$$

is finite, then so are the other two, and the identity above holds. It is then (and only then) permissible to change the order of integrations.

Fubini's theorem is of fundamental importance in the study of tensor products and convolutions of distributions.

1.3.2.2.6. *Topology in function spaces*

Geometric intuition, which often makes 'obvious' the topological properties of the real line and of ordinary space, cannot be relied upon in the study of function spaces: the latter are infinite-dimensional, and several inequivalent notions of convergence may exist. A careful analysis of topological concepts and of their interrelationship is thus a necessary prerequisite to the study of these spaces. The reader may consult Dieudonné (1969, 1970), Friedman (1970), Trèves (1967) and Yosida (1965) for detailed expositions.

1.3.2.2.6.1. *General topology*

Most topological notions are first encountered in the setting of *metric spaces*. A metric space E is a set equipped with a *distance function* d from $E \times E$ to the non-negative reals which satisfies:

(i)	$d(x, y) = d(y, x)$	$\forall x, y \in E$	(symmetry);
(ii)	$d(x, y) = 0$	iff $x = y$	(separation);
(iii)	$d(x, z) \leq d(x, y) + d(y, z)$	$\forall x, y, z \in E$	(triangular inequality).

By means of d, the following notions can be defined: open balls, neighbourhoods; open and closed sets, interior and closure; convergence of sequences, continuity of mappings; Cauchy sequences and completeness; compactness; connectedness. They suffice for the investigation of a great number of questions in analysis and geometry (see *e.g.* Dieudonné, 1969).

Many of these notions turn out to depend only on the properties of the collection $\mathcal{O}(E)$ of open subsets of E: two distance functions leading to the same $\mathcal{O}(E)$ lead to identical topological properties. An axiomatic reformulation of topological notions is thus possible: a *topology* in E is a collection $\mathcal{O}(E)$ of subsets of E which satisfy suitable axioms and are deemed open irrespective of the way they are obtained. From the practical standpoint, however, a topology which can be obtained from a distance function (called a *metrizable* topology) has the very useful property that *the notions of closure, limit and continuity may be defined by means of sequences*. For non-metrizable topologies, these notions are much more difficult to handle, requiring the use of 'filters' instead of sequences.

In some spaces E, a topology may be most naturally defined by a family of *pseudo-distances* $(d_\alpha)_{\alpha \in A}$, where each d_α satisfies (i) and (iii) but not (ii). Such spaces are called *uniformizable*. If for every pair $(x, y) \in E \times E$ there exists $\alpha \in A$ such that $d_\alpha(x, y) \neq 0$, then the separation property can be recovered. If furthermore a *countable* subfamily of the d_α suffices to define the topology of E, the latter can be shown to be *metrizable*, so that limiting processes in E may be studied by means of sequences.

1.3.2.2.6.2. *Topological vector spaces*

The function spaces E of interest in Fourier analysis have an underlying vector space structure over the field $\mathbb{C}$ of complex numbers. A topology on E is said to be *compatible* with a vector space structure on E if vector addition [*i.e.* the map $(\mathbf{x}, \mathbf{y}) \longmapsto \mathbf{x} + \mathbf{y}$] and scalar multiplication [*i.e.* the map $(\lambda, \mathbf{x}) \longmapsto \lambda\mathbf{x}$] are both *continuous*; E is then called a *topological vector space*. Such a topology may be defined by specifying a 'fundamental system S of neighbourhoods of $\mathbf{0}$', which can then be translated by vector addition to construct neighbourhoods of other points $\mathbf{x} \neq \mathbf{0}$.

A *norm* ν on a vector space E is a non-negative real-valued function on $E \times E$ such that

(i′)	$\nu(\lambda\mathbf{x}) = |\lambda|\nu(\mathbf{x})$	for all $\lambda \in \mathbb{C}$ and $\mathbf{x} \in E$;
(ii′)	$\nu(\mathbf{x}) = 0$	if and only if $\mathbf{x} = \mathbf{0}$;
(iii′)	$\nu(\mathbf{x} + \mathbf{y}) \leq \nu(\mathbf{x}) + \nu(\mathbf{y})$	for all $\mathbf{x}, \mathbf{y} \in E$.

Subsets of E defined by conditions of the form $\nu(\mathbf{x}) \leq r$ with $r > 0$ form a fundamental system of neighbourhoods of $\mathbf{0}$. The corresponding topology makes E a *normed space*. This topology is *metrizable*, since it is equivalent to that derived from the translation-invariant distance $d(\mathbf{x}, \mathbf{y}) = \nu(\mathbf{x} - \mathbf{y})$. Normed spaces which are *complete*, *i.e.* in which all Cauchy sequences converge, are called *Banach spaces*; they constitute the natural setting for the study of differential calculus.

A *semi-norm* σ on a vector space E is a positive real-valued function on $E \times E$ which satisfies (i′) and (iii′) but not (ii′). Given a set Σ of semi-norms on E such that any pair $(\mathbf{x}, \mathbf{y})$ in $E \times E$ is separated by at least one $\sigma \in \Sigma$, let B be the set of those subsets $\Gamma_{\sigma, r}$ of E defined by a condition of the form $\sigma(\mathbf{x}) \leq r$ with $\sigma \in \Sigma$ and $r > 0$; and let S be the set of finite intersections of elements of B. Then there exists a unique topology on E for which S is a fundamental system of neighbourhoods of $\mathbf{0}$. This topology is *uniformizable* since it is equivalent to that derived from the family of translation-invariant pseudo-distances $(\mathbf{x}, \mathbf{y}) \longmapsto \sigma(\mathbf{x} - \mathbf{y})$. It is *metrizable* if and only if it can be constructed by the above procedure with Σ a *countable* set of semi-norms. If furthermore E is complete, E is called a *Fréchet space*.

If E is a topological vector space over $\mathbb{C}$, its *dual* E^* is the set of all linear mappings from E to $\mathbb{C}$ (which are also called *linear forms*, or *linear functionals*, over E). The subspace of E^* consisting of all linear forms which are *continuous* for the topology of E is called the *topological dual* of E and is denoted E'. If the topology on E is metrizable, then the continuity of a linear form $T \in E'$ at $f \in E$ can be ascertained by means of sequences, *i.e.* by checking that the sequence $[T(f_j)]$ of complex numbers converges to $T(f)$ in $\mathbb{C}$ whenever the sequence (f_j) converges to f in E.

1.3.2.3. *Elements of the theory of distributions*

1.3.2.3.1. *Origins*

At the end of the 19th century, Heaviside proposed under the name of 'operational calculus' a set of rules for solving a class of

references