

1. GENERAL RELATIONSHIPS AND TECHNIQUES

fundamental system  $S$  of neighbourhoods of the origin in  $\mathcal{E}(\Omega)$  is given by subsets of  $\mathcal{E}(\Omega)$  of the form

$$V(m, \varepsilon, K) = \{\varphi \in \mathcal{E}(\Omega) \mid |\mathbf{p}| \leq m \Rightarrow \sigma_{\mathbf{p}, K}(\varphi) < \varepsilon\}$$

for all natural integers  $m$ , positive real  $\varepsilon$ , and compact subset  $K$  of  $\Omega$ . Since a countable family of compact subsets  $K$  suffices to cover  $\Omega$ , and since restricted values of  $\varepsilon$  of the form  $\varepsilon = 1/N$  lead to the same topology,  $S$  is equivalent to a countable system of neighbourhoods and hence  $\mathcal{E}(\Omega)$  is metrizable.

Convergence in  $\mathcal{E}$  may thus be defined by means of sequences. A sequence  $(\varphi_\nu)$  in  $\mathcal{E}$  will be said to converge to 0 if for any given  $V(m, \varepsilon, K)$  there exists  $\nu_0$  such that  $\varphi_\nu \in V(m, \varepsilon, K)$  whenever  $\nu > \nu_0$ ; in other words, if the  $\varphi_\nu$  and all their derivatives  $D^{\mathbf{p}}\varphi_\nu$  converge to 0 uniformly on any given compact  $K$  in  $\Omega$ .

1.3.2.3.3.2. Topology on  $\mathcal{Q}_K(\Omega)$

It is defined by the family of semi-norms

$$\varphi \in \mathcal{Q}_K(\Omega) \mapsto \sigma_{\mathbf{p}}(\varphi) = \sup_{\mathbf{x} \in K} |D^{\mathbf{p}}\varphi(\mathbf{x})|,$$

where  $K$  is now fixed. The fundamental system  $S$  of neighbourhoods of the origin in  $\mathcal{Q}_K$  is given by sets of the form

$$V(m, \varepsilon) = \{\varphi \in \mathcal{Q}_K(\Omega) \mid |\mathbf{p}| \leq m \Rightarrow \sigma_{\mathbf{p}}(\varphi) < \varepsilon\}.$$

It is equivalent to the countable subsystem of the  $V(m, 1/N)$ , hence  $\mathcal{Q}_K(\Omega)$  is metrizable.

Convergence in  $\mathcal{Q}_K$  may thus be defined by means of sequences. A sequence  $(\varphi_\nu)$  in  $\mathcal{Q}_K$  will be said to converge to 0 if for any given  $V(m, \varepsilon)$  there exists  $\nu_0$  such that  $\varphi_\nu \in V(m, \varepsilon)$  whenever  $\nu > \nu_0$ ; in other words, if the  $\varphi_\nu$  and all their derivatives  $D^{\mathbf{p}}\varphi_\nu$  converge to 0 uniformly in  $K$ .

1.3.2.3.3.3. Topology on  $\mathcal{Q}(\Omega)$

It is defined by the fundamental system of neighbourhoods of the origin consisting of sets of the form

$$V((m), (\varepsilon)) = \left\{ \varphi \in \mathcal{Q}(\Omega) \mid |\mathbf{p}| \leq m_\nu \Rightarrow \sup_{\|\mathbf{x}\| \leq \nu} |D^{\mathbf{p}}\varphi(\mathbf{x})| < \varepsilon_\nu \text{ for all } \nu \right\},$$

where  $(m)$  is an increasing sequence  $(m_\nu)$  of integers tending to  $+\infty$  and  $(\varepsilon)$  is a decreasing sequence  $(\varepsilon_\nu)$  of positive reals tending to 0, as  $\nu \rightarrow \infty$ .

This topology is not metrizable, because the sets of sequences  $(m)$  and  $(\varepsilon)$  are essentially uncountable. It can, however, be shown to be the inductive limit of the topology of the subspaces  $\mathcal{Q}_K$ , in the following sense:  $V$  is a neighbourhood of the origin in  $\mathcal{Q}$  if and only if its intersection with  $\mathcal{Q}_K$  is a neighbourhood of the origin in  $\mathcal{Q}_K$  for any given compact  $K$  in  $\Omega$ .

A sequence  $(\varphi_\nu)$  in  $\mathcal{Q}$  will thus be said to converge to 0 in  $\mathcal{Q}$  if all the  $\varphi_\nu$  belong to some  $\mathcal{Q}_K$  (with  $K$  a compact subset of  $\Omega$  independent of  $\nu$ ) and if  $(\varphi_\nu)$  converges to 0 in  $\mathcal{Q}_K$ .

As a result, a complex-valued functional  $T$  on  $\mathcal{Q}$  will be said to be continuous for the topology of  $\mathcal{Q}$  if and only if, for any given compact  $K$  in  $\Omega$ , its restriction to  $\mathcal{Q}_K$  is continuous for the topology of  $\mathcal{Q}_K$ , i.e. maps convergent sequences in  $\mathcal{Q}_K$  to convergent sequences in  $\mathbb{C}$ .

This property of  $\mathcal{Q}$ , i.e. having a non-metrizable topology which is the inductive limit of metrizable topologies in its subspaces  $\mathcal{Q}_K$ , conditions the whole structure of distribution theory and dictates that of many of its proofs.

1.3.2.3.3.4. Topologies on  $\mathcal{E}^{(m)}$ ,  $\mathcal{Q}_K^{(m)}$ ,  $\mathcal{Q}^{(m)}$

These are defined similarly, but only involve conditions on derivatives up to order  $m$ .

1.3.2.3.4. Definition of distributions

A distribution  $T$  on  $\Omega$  is a linear form over  $\mathcal{Q}(\Omega)$ , i.e. a map

$$T : \varphi \mapsto \langle T, \varphi \rangle$$

which associates linearly a complex number  $\langle T, \varphi \rangle$  to any  $\varphi \in \mathcal{Q}(\Omega)$ , and which is continuous for the topology of that space. In the terminology of Section 1.3.2.2.6.2,  $T$  is an element of  $\mathcal{Q}'(\Omega)$ , the topological dual of  $\mathcal{Q}(\Omega)$ .

Continuity over  $\mathcal{Q}$  is equivalent to continuity over  $\mathcal{Q}_K$  for all compact  $K$  contained in  $\Omega$ , and hence to the condition that for any sequence  $(\varphi_\nu)$  in  $\mathcal{Q}$  such that

- (i)  $\text{Supp } \varphi_\nu$  is contained in some compact  $K$  independent of  $\nu$ ,
- (ii) the sequences  $(|D^{\mathbf{p}}\varphi_\nu|)$  converge uniformly to 0 on  $K$  for all multi-indices  $\mathbf{p}$ ;

then the sequence of complex numbers  $\langle T, \varphi_\nu \rangle$  converges to 0 in  $\mathbb{C}$ .

If the continuity of a distribution  $T$  requires (ii) for  $|\mathbf{p}| \leq m$  only,  $T$  may be defined over  $\mathcal{Q}^{(m)}$  and thus  $T \in \mathcal{Q}'^{(m)}$ ;  $T$  is said to be a distribution of finite order  $m$ . In particular, for  $m = 0$ ,  $\mathcal{Q}^{(0)}$  is the space of continuous functions with compact support, and a distribution  $T \in \mathcal{Q}'^{(0)}$  is a (Radon) measure as used in the theory of integration. Thus measures are particular cases of distributions.

Generally speaking, the larger a space of test functions, the smaller its topological dual:

$$m < n \Rightarrow \mathcal{Q}^{(m)} \supset \mathcal{Q}^{(n)} \Rightarrow \mathcal{Q}'^{(n)} \supset \mathcal{Q}'^{(m)}.$$

This clearly results from the observation that if the  $\varphi$ 's are allowed to be less regular, then less wildness can be accommodated in  $T$  if the continuity of the map  $\varphi \mapsto \langle T, \varphi \rangle$  with respect to  $\varphi$  is to be preserved.

1.3.2.3.5. First examples of distributions

(i) The linear map  $\varphi \mapsto \langle \delta, \varphi \rangle = \varphi(\mathbf{0})$  is a measure (i.e. a zeroth-order distribution) called Dirac's measure or (improperly) Dirac's 'delta-function'.

(ii) The linear map  $\varphi \mapsto \langle \delta_{(\mathbf{a})}, \varphi \rangle = \varphi(\mathbf{a})$  is called Dirac's measure at point  $\mathbf{a} \in \mathbb{R}^n$ .

(iii) The linear map  $\varphi \mapsto (-1)^{|\mathbf{p}|} D^{\mathbf{p}}\varphi(\mathbf{a})$  is a distribution of order  $m = |\mathbf{p}| > 0$ , and hence is not a measure.

(iv) The linear map  $\varphi \mapsto \sum_{\nu > 0} \varphi^{(\nu)}(\nu)$  is a distribution of infinite order on  $\mathbb{R}$ : the order of differentiation is bounded for each  $\varphi$  (because  $\varphi$  has compact support) but is not as  $\varphi$  varies.

(v) If  $(\mathbf{p}_\nu)$  is a sequence of multi-indices  $\mathbf{p}_\nu = (p_{1\nu}, \dots, p_{n\nu})$  such that  $|\mathbf{p}_\nu| \rightarrow \infty$  as  $\nu \rightarrow \infty$ , then the linear map  $\varphi \mapsto \sum_{\nu > 0} (D^{\mathbf{p}_\nu}\varphi)(\mathbf{p}_\nu)$  is a distribution of infinite order on  $\mathbb{R}^n$ .

1.3.2.3.6. Distributions associated to locally integrable functions

Let  $f$  be a complex-valued function over  $\Omega$  such that  $\int_K |f(\mathbf{x})| d^n \mathbf{x}$  exists for any given compact  $K$  in  $\Omega$ ;  $f$  is then called locally integrable.

The linear mapping from  $\mathcal{Q}(\Omega)$  to  $\mathbb{C}$  defined by

$$\varphi \mapsto \int_{\Omega} f(\mathbf{x})\varphi(\mathbf{x}) d^n \mathbf{x}$$

may then be shown to be continuous over  $\mathcal{Q}(\Omega)$ . It thus defines a distribution  $T_f \in \mathcal{Q}'(\Omega)$ :

$$\langle T_f, \varphi \rangle = \int_{\Omega} f(\mathbf{x})\varphi(\mathbf{x}) d^n \mathbf{x}.$$

As the continuity of  $T_f$  only requires that  $\varphi \in \mathcal{Q}^{(0)}(\Omega)$ ,  $T_f$  is actually a Radon measure.