International Tables for Crystallography (2006). Vol. B, Section 1.3.2.3.6, pp. 30–31.

fundamental system S of neighbourhoods of the origin in $\mathcal{E}(\Omega)$ is given by subsets of $\mathcal{E}(\Omega)$ of the form

$$V(m, \varepsilon, K) = \{\varphi \in \mathcal{E}(\Omega) | |\mathbf{p}| \leq m \Rightarrow \sigma_{\mathbf{p},K}(\varphi) < \varepsilon\}$$

for all natural integers m, positive real ε, and compact subset K of Ω. Since a *countable* family of compact subsets K suffices to cover Ω, and since restricted values of ε of the form $\varepsilon = 1/N$ lead to the same topology, S is equivalent to a *countable* system of neighbourhoods and hence $\mathcal{E}(\Omega)$ *is metrizable*.

Convergence in $\mathcal{E}$ may thus be defined by means of sequences. A sequence (φ_ν) in $\mathcal{E}$ will be said to converge to 0 if for any given $V(m, \varepsilon, K)$ there exists ν_0 such that $\varphi_\nu \in V(m, \varepsilon, K)$ whenever $\nu > \nu_0$; in other words, if the φ_ν and all their derivatives $D^{\mathbf{p}}\varphi_\nu$ converge to 0 uniformly on any given compact K in Ω.

1.3.2.3.3.2. *Topology on* $\mathcal{D}_k(\Omega)$

It is defined by the family of semi-norms

$$\varphi \in \mathcal{D}_K(\Omega) \longmapsto \sigma_{\mathbf{p}}(\varphi) = \sup_{\mathbf{x} \in K} |D^{\mathbf{p}}\varphi(\mathbf{x})|,$$

where K is now fixed. The fundamental system S of neighbourhoods of the origin in $\mathcal{D}_K$ is given by sets of the form

$$V(m, \varepsilon) = \{\varphi \in \mathcal{D}_K(\Omega) | |\mathbf{p}| \leq m \Rightarrow \sigma_{\mathbf{p}}(\varphi) < \varepsilon\}.$$

It is equivalent to the countable subsystem of the $V(m, 1/N)$, hence $\mathcal{D}_K(\Omega)$ *is metrizable*.

Convergence in $\mathcal{D}_K$ may thus be defined by means of sequences. A sequence (φ_ν) in $\mathcal{D}_K$ will be said to converge to 0 if for any given $V(m, \varepsilon)$ there exists ν_0 such that $\varphi_\nu \in V(m, \varepsilon)$ whenever $\nu > \nu_0$; in other words, if the φ_ν and all their derivatives $D^{\mathbf{p}}\varphi_\nu$ converge to 0 uniformly in K.

1.3.2.3.3.3. *Topology on* $\mathcal{D}(\Omega)$

It is defined by the fundamental system of neighbourhoods of the origin consisting of sets of the form

$$V((m), (\varepsilon)) = \left\{\varphi \in \mathcal{D}(\Omega) | |\mathbf{p}| \leq m_\nu \Rightarrow \sup_{\|\mathbf{x}\| \leq \nu} |D^{\mathbf{p}}\varphi(\mathbf{x})| < \varepsilon_\nu \text{ for all } \nu\right\},$$

where (m) is an increasing sequence (m_ν) of integers tending to $+\infty$ and (ε) is a decreasing sequence (ε_ν) of positive reals tending to 0, as $\nu \to \infty$.

This topology is *not metrizable*, because the sets of sequences (m) and (ε) are essentially uncountable. It can, however, be shown to be the *inductive limit* of the topology of the subspaces $\mathcal{D}_K$, in the following sense: V is a neighbourhood of the origin in $\mathcal{D}$ if and only if its intersection with $\mathcal{D}_K$ is a neighbourhood of the origin in $\mathcal{D}_K$ for any given compact K in Ω.

A sequence (φ_ν) in $\mathcal{D}$ will thus be said to converge to 0 in $\mathcal{D}$ if all the φ_ν belong to some $\mathcal{D}_K$ (with K a compact subset of Ω independent of ν) and if (φ_ν) converges to 0 in $\mathcal{D}_K$.

As a result, a complex-valued functional T on $\mathcal{D}$ will be said to be continuous for the topology of $\mathcal{D}$ if and only if, for any given compact K in Ω, its restriction to $\mathcal{D}_K$ is continuous for the topology of $\mathcal{D}_K$, *i.e.* maps convergent sequences in $\mathcal{D}_K$ to convergent sequences in $\mathbb{C}$.

This property of $\mathcal{D}$, *i.e.* having a non-metrizable topology which is the inductive limit of metrizable topologies in its subspaces $\mathcal{D}_K$, conditions the whole structure of distribution theory and dictates that of many of its proofs.

1.3.2.3.3.4. *Topologies on* $\mathcal{E}^{(m)}, \mathcal{D}_k^{(m)}, \mathcal{D}^{(m)}$

These are defined similarly, but only involve conditions on derivatives up to order m.

1.3.2.3.4. *Definition of distributions*

A *distribution* T on Ω is a *linear form* over $\mathcal{D}(\Omega)$, *i.e.* a map

$$T : \varphi \longmapsto \langle T, \varphi \rangle$$

which associates linearly a complex number $\langle T, \varphi \rangle$ to any $\varphi \in \mathcal{D}(\Omega)$, and which is *continuous* for the topology of that space. In the terminology of Section 1.3.2.2.6.2, T is an element of $\mathcal{D}'(\Omega)$, the *topological dual* of $\mathcal{D}(\Omega)$.

Continuity over $\mathcal{D}$ is equivalent to continuity over $\mathcal{D}_K$ for all compact K contained in Ω, and hence to the condition that for any sequence (φ_ν) in $\mathcal{D}$ such that

(i) Supp φ_ν is contained in some compact K independent of ν,

(ii) the sequences $(|D^{\mathbf{p}}\varphi_\nu|)$ converge uniformly to 0 on K for all multi-indices $\mathbf{p}$;

then the sequence of complex numbers $\langle T, \varphi_\nu \rangle$ converges to 0 in $\mathbb{C}$.

If the continuity of a distribution T requires (ii) for $|\mathbf{p}| \leq m$ only, T may be defined over $\mathcal{D}^{(m)}$ and thus $T \in \mathcal{D}'^{(m)}$; T is said to be a *distribution of finite order m*. In particular, for $m = 0$, $\mathcal{D}^{(0)}$ is the space of continuous functions with compact support, and a distribution $T \in \mathcal{D}'^{(0)}$ is a (Radon) *measure* as used in the theory of integration. Thus measures are particular cases of distributions.

Generally speaking, the *larger* a space of test functions, the *smaller* its topological dual:

$$m < n \Rightarrow \mathcal{D}^{(m)} \supset \mathcal{D}^{(n)} \Rightarrow \mathcal{D}'^{(n)} \supset \mathcal{D}'^{(m)}.$$

This clearly results from the observation that if the φ's are allowed to be less regular, then less wildness can be accommodated in T if the continuity of the map $\varphi \longmapsto \langle T, \varphi \rangle$ with respect to φ is to be preserved.

1.3.2.3.5. *First examples of distributions*

(i) The linear map $\varphi \longmapsto \langle \delta, \varphi \rangle = \varphi(\mathbf{0})$ is a measure (*i.e.* a zeroth-order distribution) called Dirac's measure or (improperly) Dirac's 'δ-function'.

(ii) The linear map $\varphi \longmapsto \langle \delta_{(\mathbf{a})}, \varphi \rangle = \varphi(\mathbf{a})$ is called Dirac's measure at point $\mathbf{a} \in \mathbb{R}^n$.

(iii) The linear map $\varphi \longmapsto (-1)^{\mathbf{p}} D^{\mathbf{p}}\varphi(\mathbf{a})$ is a distribution of order $m = |\mathbf{p}| > 0$, and hence is not a measure.

(iv) The linear map $\varphi \longmapsto \sum_{\nu > 0} \varphi^{(\nu)}(\nu)$ is a distribution of infinite order on $\mathbb{R}$: the order of differentiation is bounded for each φ (because φ has compact support) but is not as φ varies.

(v) If $(\mathbf{p}_\nu)$ is a sequence of multi-indices $\mathbf{p}_\nu = (p_{1\nu}, \ldots, p_{n\nu})$ such that $|\mathbf{p}_\nu| \to \infty$ as $\nu \to \infty$, then the linear map $\varphi \longmapsto \sum_{\nu > 0} (D^{\mathbf{p}_\nu}\varphi)(\mathbf{p}_\nu)$ is a distribution of infinite order on $\mathbb{R}^n$.

1.3.2.3.6. *Distributions associated to locally integrable functions*

Let f be a complex-valued function over Ω such that $\int_K |f(\mathbf{x})| \,\mathrm{d}^n\mathbf{x}$ exists for any given compact K in Ω; f is then called *locally integrable*.

The linear mapping from $\mathcal{D}(\Omega)$ to $\mathbb{C}$ defined by

$$\varphi \longmapsto \int_\Omega f(\mathbf{x})\varphi(\mathbf{x}) \,\mathrm{d}^n\mathbf{x}$$

may then be shown to be continuous over $\mathcal{D}(\Omega)$. It thus defines a *distribution* $T_f \in \mathcal{D}'(\Omega)$:

$$\langle T_f, \varphi \rangle = \int_\Omega f(\mathbf{x})\varphi(\mathbf{x}) \,\mathrm{d}^n\mathbf{x}.$$

As the continuity of T_f only requires that $\varphi \in \mathcal{D}^{(0)}(\Omega)$, T_f is actually a Radon measure.

It can be shown that two locally integrable functions f and g define the same distribution, *i.e.*

$$\langle T_f, \varphi\rangle = \langle T_K, \varphi\rangle \quad \text{for all } \varphi \in \mathscr{D},$$

if and only if they are equal almost everywhere. The classes of locally integrable functions modulo this equivalence form a vector space denoted $L^1_{\rm loc}(\Omega)$; each element of $L^1_{\rm loc}(\Omega)$ may therefore be identified with the distribution T_f defined by any one of its representatives f.

1.3.2.3.7. *Support of a distribution*

A distribution $T \in \mathscr{D}'(\Omega)$ is said to *vanish* on an open subset ω of Ω if it vanishes on all functions in $\mathscr{D}(\omega)$, *i.e.* if $\langle T, \varphi\rangle = 0$ whenever $\varphi \in \mathscr{D}(\omega)$.

The *support* of a distribution T, denoted Supp T, is then defined as the complement of the set-theoretic union of those open subsets ω on which T vanishes; or equivalently as the smallest closed subset of Ω outside which T vanishes.

When $T = T_f$ for $f \in L^1_{\rm loc}(\Omega)$, then Supp T = Supp f, so that the two notions coincide. Clearly, if Supp T and Supp φ are disjoint subsets of Ω, then $\langle T, \varphi\rangle = 0$.

It can be shown that any distribution $T \in \mathscr{D}'$ with compact support may be extended from $\mathscr{D}$ to $\mathscr{E}$ while remaining continuous, so that $T \in \mathscr{E}'$; and that conversely, if $S \in \mathscr{E}'$, then its restriction T to $\mathscr{D}$ is a distribution with compact support. Thus, *the topological dual $\mathscr{E}'$ of $\mathscr{E}$ consists of those distributions in $\mathscr{D}'$ which have compact support*. This is intuitively clear since, if the condition of having compact support is fulfilled by T, it needs no longer be required of φ, which may then roam through $\mathscr{E}$ rather than $\mathscr{D}$.

1.3.2.3.8. *Convergence of distributions*

A *sequence* (T_j) of distributions will be said to converge in $\mathscr{D}'$ to a distribution T as $j \to \infty$ if, for any given $\varphi \in \mathscr{D}$, the sequence of complex numbers $(\langle T_j, \varphi\rangle)$ converges in $\mathbb{C}$ to the complex number $\langle T, \varphi\rangle$.

A *series* $\sum_{j=0}^{\infty} T_j$ of distributions will be said to converge in $\mathscr{D}'$ and to have distribution S as its sum if the sequence of partial sums $S_k = \sum_{j=0}^{k}$ converges to S.

These definitions of convergence in $\mathscr{D}'$ assume that the limits T and S are known in advance, and are distributions. This raises the question of the *completeness* of $\mathscr{D}'$: if a sequence (T_j) in $\mathscr{D}'$ is such that the sequence $(\langle T_j, \varphi\rangle)$ has a limit in $\mathbb{C}$ for all $\varphi \in \mathscr{D}$, does the map

$$\varphi \longmapsto \lim_{j\to\infty} \langle T_j, \varphi\rangle$$

define a distribution $T \in \mathscr{D}'$? In other words, does the limiting process preserve continuity with respect to φ? It is a remarkable theorem that, because of the strong topology on $\mathscr{D}$, this is actually the case. An analogous statement holds for series. This notion of convergence does not coincide with any of the classical notions used for ordinary functions: for example, the sequence (φ_ν) with $\varphi_\nu(x) = \cos \nu x$ converges to 0 in $\mathscr{D}'(\mathbb{R})$, but fails to do so by any of the standard criteria.

An example of convergent sequences of distributions is provided by sequences which converge to δ. If (f_ν) is a sequence of locally summable functions on $\mathbb{R}^n$ such that

(i) $\int_{\|\mathbf{x}\|< b} f_\nu(\mathbf{x}) \,{\rm d}^n\mathbf{x} \to 1$ as $\nu \to \infty$ for all $b > 0$;

(ii) $\int_{a\le\|\mathbf{x}\|\le 1/a} |f_\nu(\mathbf{x})| \,{\rm d}^n\mathbf{x} \to 0$ as $\nu \to \infty$ for all $0 < a < 1$;

(iii) there exists $d > 0$ and $M > 0$ such that $\int_{\|\mathbf{x}\|< d} |f_\nu(\mathbf{x})| \,{\rm d}^n\mathbf{x} < M$ for all ν;

then the sequence (T_{f_ν}) of distributions converges to δ in $\mathscr{D}'(\mathbb{R}^n)$.

1.3.2.3.9. *Operations on distributions*

As a general rule, the definitions are chosen so that the operations coincide with those on functions whenever a distribution is associated to a function.

Most definitions consist in transferring to a distribution T an operation which is well defined on $\varphi \in \mathscr{D}$ by 'transposing' it in the duality product $\langle T, \varphi\rangle$; this procedure will map T to a new distribution provided the original operation maps $\mathscr{D}$ *continuously* into itself.

1.3.2.3.9.1. *Differentiation*

(*a*) *Definition and elementary properties*

If T is a distribution on $\mathbb{R}^n$, its partial derivative $\partial_i T$ with respect to x_i is defined by

$$\langle \partial_i T, \varphi\rangle = -\langle T, \partial_i \varphi\rangle$$

for all $\varphi \in \mathscr{D}$. This does define a distribution, because the partial differentiations $\varphi \longmapsto \partial_i\varphi$ are continuous for the topology of $\mathscr{D}$.

Suppose that $T = T_f$ with f a locally integrable function such that $\partial_i f$ exists and is almost everywhere continuous. Then integration by parts along the x_i axis gives

$$\begin{aligned}
&\int_{\mathbb{R}^n} \partial_i f(x_1, \ldots, x_i, \ldots, x_n)\varphi(x_1, \ldots, x_i, \ldots, x_n) \,{\rm d}x_i \\
&\quad = (f\varphi)(x_1, \ldots, +\infty, \ldots, x_n) - (f\varphi)(x_1, \ldots, -\infty, \ldots, x_n) \\
&\qquad - \int_{\mathbb{R}^n} f(x_1, \ldots, x_i, \ldots, x_n)\partial_i\varphi(x_1, \ldots, x_i, \ldots, x_n) \,{\rm d}x_i;
\end{aligned}$$

the integrated term vanishes, since φ has compact support, showing that $\partial_i T_f = T_{\partial_i f}$.

The test functions $\varphi \in \mathscr{D}$ are infinitely differentiable. Therefore, transpositions like that used to define $\partial_i T$ may be repeated, so that *any distribution is infinitely differentiable*. For instance,

$$\begin{aligned}
\langle \partial^2_{ij} T, \varphi\rangle &= -\langle \partial_j T, \partial_i\varphi\rangle = \langle T, \partial^2_{ij}\varphi\rangle, \\
\langle D^{\mathbf{p}} T, \varphi\rangle &= (-1)^{|\mathbf{p}|}\langle T, D^{\mathbf{p}}\varphi\rangle, \\
\langle \Delta T, \varphi\rangle &= \langle T, \Delta\varphi\rangle,
\end{aligned}$$

where Δ is the Laplacian operator. The derivatives of Dirac's δ distribution are

$$\langle D^{\mathbf{p}}\delta, \varphi\rangle = (-1)^{|\mathbf{p}|}\langle \delta, D^{\mathbf{p}}\varphi\rangle = (-1)^{|\mathbf{p}|} D^{\mathbf{p}}\varphi(\mathbf{0}).$$

It is remarkable that *differentiation is a continuous operation* for the topology on $\mathscr{D}'$: if a sequence (T_j) of distributions converges to distribution T, then the sequence $(D^{\mathbf{p}} T_j)$ of derivatives converges to $D^{\mathbf{p}} T$ for any multi-index $\mathbf{p}$, since as $j \to \infty$

$$\langle D^{\mathbf{p}} T_j, \varphi\rangle = (-1)^{|\mathbf{p}|}\langle T_j, D^{\mathbf{p}}\varphi\rangle \to (-1)^{|\mathbf{p}|}\langle T, D^{\mathbf{p}}\varphi\rangle = \langle D^{\mathbf{p}} T, \varphi\rangle.$$

An analogous statement holds for series: any convergent series of distributions may be differentiated termwise to all orders. This illustrates how 'robust' the constructs of distribution theory are in comparison with those of ordinary function theory, where similar statements are notoriously untrue.

(*b*) *Differentiation under the duality bracket*

Limiting processes and differentiation may also be carried out under the duality bracket $\langle , \rangle$ as under the integral sign with ordinary functions. Let the function $\varphi = \varphi(\mathbf{x}, \lambda)$ depend on a parameter $\lambda \in \Lambda$ and a vector $\mathbf{x} \in \mathbb{R}^n$ in such a way that all functions

$$\varphi_\lambda : \mathbf{x} \longmapsto \varphi(\mathbf{x}, \lambda)$$

be in $\mathscr{D}(\mathbb{R}^n)$ for all $\lambda \in \Lambda$. Let $T \in \mathscr{D}'(\mathbb{R}^n)$ be a distribution, let

$$I(\lambda) = \langle T, \varphi_\lambda\rangle$$

references