

1.3. FOURIER TRANSFORMS IN CRYSTALLOGRAPHY

It can be shown that two locally integrable functions  $f$  and  $g$  define the same distribution, i.e.

$$\langle T_f, \varphi \rangle = \langle T_g, \varphi \rangle \quad \text{for all } \varphi \in \mathcal{D},$$

if and only if they are equal almost everywhere. The classes of locally integrable functions modulo this equivalence form a vector space denoted  $L^1_{\text{loc}}(\Omega)$ ; each element of  $L^1_{\text{loc}}(\Omega)$  may therefore be identified with the distribution  $T_f$  defined by any one of its representatives  $f$ .

1.3.2.3.7. Support of a distribution

A distribution  $T \in \mathcal{D}'(\Omega)$  is said to *vanish* on an open subset  $\omega$  of  $\Omega$  if it vanishes on all functions in  $\mathcal{D}(\omega)$ , i.e. if  $\langle T, \varphi \rangle = 0$  whenever  $\varphi \in \mathcal{D}(\omega)$ .

The *support* of a distribution  $T$ , denoted  $\text{Supp } T$ , is then defined as the complement of the set-theoretic union of those open subsets  $\omega$  on which  $T$  vanishes; or equivalently as the smallest closed subset of  $\Omega$  outside which  $T$  vanishes.

When  $T = T_f$  for  $f \in L^1_{\text{loc}}(\Omega)$ , then  $\text{Supp } T = \text{Supp } f$ , so that the two notions coincide. Clearly, if  $\text{Supp } T$  and  $\text{Supp } \varphi$  are disjoint subsets of  $\Omega$ , then  $\langle T, \varphi \rangle = 0$ .

It can be shown that any distribution  $T \in \mathcal{D}'$  with compact support may be extended from  $\mathcal{D}$  to  $\mathcal{E}$  while remaining continuous, so that  $T \in \mathcal{E}'$ ; and that conversely, if  $S \in \mathcal{E}'$ , then its restriction  $T$  to  $\mathcal{D}$  is a distribution with compact support. Thus, *the topological dual  $\mathcal{E}'$  of  $\mathcal{E}$  consists of those distributions in  $\mathcal{D}'$  which have compact support*. This is intuitively clear since, if the condition of having compact support is fulfilled by  $T$ , it needs no longer be required of  $\varphi$ , which may then roam through  $\mathcal{E}$  rather than  $\mathcal{D}$ .

1.3.2.3.8. Convergence of distributions

A sequence  $(T_j)$  of distributions will be said to converge in  $\mathcal{D}'$  to a distribution  $T$  as  $j \rightarrow \infty$  if, for any given  $\varphi \in \mathcal{D}$ , the sequence of complex numbers  $(\langle T_j, \varphi \rangle)$  converges in  $\mathbb{C}$  to the complex number  $\langle T, \varphi \rangle$ .

A series  $\sum_{j=0}^{\infty} T_j$  of distributions will be said to converge in  $\mathcal{D}'$  and to have distribution  $S$  as its sum if the sequence of partial sums  $S_k = \sum_{j=0}^k T_j$  converges to  $S$ .

These definitions of convergence in  $\mathcal{D}'$  assume that the limits  $T$  and  $S$  are known in advance, and are distributions. This raises the question of the *completeness* of  $\mathcal{D}'$ : if a sequence  $(T_j)$  in  $\mathcal{D}'$  is such that the sequence  $(\langle T_j, \varphi \rangle)$  has a limit in  $\mathbb{C}$  for all  $\varphi \in \mathcal{D}$ , does the map

$$\varphi \mapsto \lim_{j \rightarrow \infty} \langle T_j, \varphi \rangle$$

define a distribution  $T \in \mathcal{D}'$ ? In other words, does the limiting process preserve continuity with respect to  $\varphi$ ? It is a remarkable theorem that, because of the strong topology on  $\mathcal{D}$ , this is actually the case. An analogous statement holds for series. This notion of convergence does not coincide with any of the classical notions used for ordinary functions: for example, the sequence  $(\varphi_\nu)$  with  $\varphi_\nu(x) = \cos \nu x$  converges to 0 in  $\mathcal{D}'(\mathbb{R})$ , but fails to do so by any of the standard criteria.

An example of convergent sequences of distributions is provided by sequences which converge to  $\delta$ . If  $(f_\nu)$  is a sequence of locally summable functions on  $\mathbb{R}^n$  such that

- (i)  $\int_{\|\mathbf{x}\| < b} f_\nu(\mathbf{x}) \, d^n \mathbf{x} \rightarrow 1$  as  $\nu \rightarrow \infty$  for all  $b > 0$ ;
- (ii)  $\int_{a \leq \|\mathbf{x}\| \leq 1/a} f_\nu(\mathbf{x}) \, d^n \mathbf{x} \rightarrow 0$  as  $\nu \rightarrow \infty$  for all  $0 < a < 1$ ;
- (iii) there exists  $d > 0$  and  $M > 0$  such that  $\int_{\|\mathbf{x}\| < d} f_\nu(\mathbf{x}) \, d^n \mathbf{x} < M$  for all  $\nu$ ;

then the sequence  $(T_{f_\nu})$  of distributions converges to  $\delta$  in  $\mathcal{D}'(\mathbb{R}^n)$ .

1.3.2.3.9. Operations on distributions

As a general rule, the definitions are chosen so that the operations coincide with those on functions whenever a distribution is associated to a function.

Most definitions consist in transferring to a distribution  $T$  an operation which is well defined on  $\varphi \in \mathcal{D}$  by ‘transposing’ it in the duality product  $\langle T, \varphi \rangle$ ; this procedure will map  $T$  to a new distribution provided the original operation maps  $\mathcal{D}$  continuously into itself.

1.3.2.3.9.1. Differentiation

(a) Definition and elementary properties

If  $T$  is a distribution on  $\mathbb{R}^n$ , its partial derivative  $\partial_i T$  with respect to  $x_i$  is defined by

$$\langle \partial_i T, \varphi \rangle = -\langle T, \partial_i \varphi \rangle$$

for all  $\varphi \in \mathcal{D}$ . This does define a distribution, because the partial differentiations  $\varphi \mapsto \partial_i \varphi$  are continuous for the topology of  $\mathcal{D}$ .

Suppose that  $T = T_f$  with  $f$  a locally integrable function such that  $\partial_i f$  exists and is almost everywhere continuous. Then integration by parts along the  $x_i$  axis gives

$$\begin{aligned} \int_{\mathbb{R}^n} \partial_i f(x_1, \dots, x_i, \dots, x_n) \varphi(x_1, \dots, x_i, \dots, x_n) \, dx_i \\ = (f\varphi)(x_1, \dots, +\infty, \dots, x_n) - (f\varphi)(x_1, \dots, -\infty, \dots, x_n) \\ - \int_{\mathbb{R}^n} f(x_1, \dots, x_i, \dots, x_n) \partial_i \varphi(x_1, \dots, x_i, \dots, x_n) \, dx_i; \end{aligned}$$

the integrated term vanishes, since  $\varphi$  has compact support, showing that  $\partial_i T_f = T_{\partial_i f}$ .

The test functions  $\varphi \in \mathcal{D}$  are infinitely differentiable. Therefore, transpositions like that used to define  $\partial_i T$  may be repeated, so that *any distribution is infinitely differentiable*. For instance,

$$\begin{aligned} \langle \partial_{ij}^2 T, \varphi \rangle &= -\langle \partial_j T, \partial_i \varphi \rangle = \langle T, \partial_{ij}^2 \varphi \rangle, \\ \langle D^{\mathbf{p}} T, \varphi \rangle &= (-1)^{|\mathbf{p}|} \langle T, D^{\mathbf{p}} \varphi \rangle, \\ \langle \Delta T, \varphi \rangle &= \langle T, \Delta \varphi \rangle, \end{aligned}$$

where  $\Delta$  is the Laplacian operator. The derivatives of Dirac’s  $\delta$  distribution are

$$\langle D^{\mathbf{p}} \delta, \varphi \rangle = (-1)^{|\mathbf{p}|} \langle \delta, D^{\mathbf{p}} \varphi \rangle = (-1)^{|\mathbf{p}|} D^{\mathbf{p}} \varphi(\mathbf{0}).$$

It is remarkable that *differentiation is a continuous operation* for the topology on  $\mathcal{D}'$ : if a sequence  $(T_j)$  of distributions converges to distribution  $T$ , then the sequence  $(D^{\mathbf{p}} T_j)$  of derivatives converges to  $D^{\mathbf{p}} T$  for any multi-index  $\mathbf{p}$ , since as  $j \rightarrow \infty$

$$\langle D^{\mathbf{p}} T_j, \varphi \rangle = (-1)^{|\mathbf{p}|} \langle T_j, D^{\mathbf{p}} \varphi \rangle \rightarrow (-1)^{|\mathbf{p}|} \langle T, D^{\mathbf{p}} \varphi \rangle = \langle D^{\mathbf{p}} T, \varphi \rangle.$$

An analogous statement holds for series: any convergent series of distributions may be differentiated termwise to all orders. This illustrates how ‘robust’ the constructs of distribution theory are in comparison with those of ordinary function theory, where similar statements are notoriously untrue.

(b) Differentiation under the duality bracket

Limiting processes and differentiation may also be carried out under the duality bracket  $\langle \cdot, \cdot \rangle$  as under the integral sign with ordinary functions. Let the function  $\varphi = \varphi(\mathbf{x}, \lambda)$  depend on a parameter  $\lambda \in \Lambda$  and a vector  $\mathbf{x} \in \mathbb{R}^n$  in such a way that all functions

$$\varphi_\lambda : \mathbf{x} \mapsto \varphi(\mathbf{x}, \lambda)$$

be in  $\mathcal{D}(\mathbb{R}^n)$  for all  $\lambda \in \Lambda$ . Let  $T \in \mathcal{D}'(\mathbb{R}^n)$  be a distribution, let

$$I(\lambda) = \langle T, \varphi_\lambda \rangle$$