$$\mathcal{F}[f'](\xi) = (2\pi i\xi)\mathcal{F}[f](\xi)$$

with the bound

$$\|2\pi\xi\mathcal{F}[f]\|_\infty \leq \|f'\|_1$$

so that $|\mathcal{F}[f](\xi)|$ decreases faster than $1/|\xi| \to \infty$.

This result can be easily extended to several dimensions and to any multi-index $\mathbf{m}$: if $f$ is summable and has continuous summable partial derivatives up to order $|\mathbf{m}|$, then

$$\mathcal{F}[D^{\mathbf{m}}f](\boldsymbol{\xi}) = (2\pi i\boldsymbol{\xi})^{\mathbf{m}}\mathcal{F}[f](\boldsymbol{\xi})$$

and

$$\|(2\pi\boldsymbol{\xi})^{\mathbf{m}}\mathcal{F}[f]\|_\infty \leq \|D^{\mathbf{m}}f\|_1.$$

Similar results hold for $\bar{\mathcal{F}}$, with $2\pi i\boldsymbol{\xi}$ replaced by $-2\pi i\boldsymbol{\xi}$. Thus, the more differentiable $f$ is, with summable derivatives, the faster $\mathcal{F}[f]$ and $\bar{\mathcal{F}}[f]$ decrease at infinity.

The property of turning differentiation into multiplication by a monomial has many important applications in crystallography, for instance differential syntheses (Sections 1.3.4.2.1.9, 1.3.4.4.7.2, 1.3.4.4.7.5) and moment-generating functions [Section 1.3.4.5.2.1(*c*)].

1.3.2.4.2.9. *Decrease at infinity*

Conversely, assume that $f$ is summable on $\mathbb{R}^n$ and that $f$ decreases fast enough at infinity for $\mathbf{x}^{\mathbf{m}}f$ also to be summable, for some multi-index $\mathbf{m}$. Then the integral defining $\mathcal{F}[f]$ may be subjected to the differential operator $D^{\mathbf{m}}$, still yielding a convergent integral: therefore $D^{\mathbf{m}}\mathcal{F}[f]$ exists, and

$$D^{\mathbf{m}}(\mathcal{F}[f])(\boldsymbol{\xi}) = \mathcal{F}[(-2\pi i\mathbf{x})^{\mathbf{m}}f](\boldsymbol{\xi})$$

with the bound

$$\|D^{\mathbf{m}}(\mathcal{F}[f])\|_\infty = \|(2\pi\mathbf{x})^{\mathbf{m}}f\|_1.$$

Similar results hold for $\bar{\mathcal{F}}$, with $-2\pi i\mathbf{x}$ replaced by $2\pi i\mathbf{x}$. Thus, the faster $f$ decreases at infinity, the more $\mathcal{F}[f]$ and $\bar{\mathcal{F}}[f]$ are differentiable, with bounded derivatives. This property is the converse of that described in Section 1.3.2.4.2.8, and their combination is fundamental in the definition of the function space $\mathcal{S}$ in Section 1.3.2.4.4.1, of tempered distributions in Section 1.3.2.5, and in the extension of the Fourier transformation to them.

1.3.2.4.2.10. *The Paley–Wiener theorem*

An extreme case of the last instance occurs when $f$ has *compact support*: then $\mathcal{F}[f]$ and $\bar{\mathcal{F}}[f]$ are so regular that they may be analytically continued from $\mathbb{R}^n$ to $\mathbb{C}^n$ where they are *entire* functions, *i.e.* have no singularities at finite distance (Paley & Wiener, 1934). This is easily seen for $\mathcal{F}[f]$: giving vector $\boldsymbol{\xi} \in \mathbb{R}^n$ a vector $\boldsymbol{\eta} \in \mathbb{R}^n$ of imaginary parts leads to

$$\begin{aligned}\mathcal{F}[f](\boldsymbol{\xi} + i\boldsymbol{\eta}) &= \int_{\mathbb{R}^n} f(\mathbf{x}) \exp[-2\pi i(\boldsymbol{\xi} + i\boldsymbol{\eta}) \cdot \mathbf{x}] \,\mathrm{d}^n\mathbf{x} \\ &= \mathcal{F}[\exp(2\pi\boldsymbol{\eta} \cdot \mathbf{x})f](\boldsymbol{\xi}),\end{aligned}$$

where the latter transform always exists since $\exp(2\pi\boldsymbol{\eta} \cdot \mathbf{x})f$ is summable with respect to $\mathbf{x}$ for all values of $\boldsymbol{\eta}$. This analytic continuation forms the basis of the saddlepoint method in probability theory [Section 1.3.4.5.2.1(*f*)] and leads to the use of maximum-entropy distributions in the statistical theory of direct phase determination [Section 1.3.4.5.2.2(*e*)].

By Liouville's theorem, an entire function in $\mathbb{C}^n$ cannot vanish identically on the complement of a compact subset of $\mathbb{R}^n$ without vanishing everywhere: therefore $\mathcal{F}[f]$ cannot have compact support if $f$ has, and hence $\mathcal{D}(\mathbb{R}^n)$ *is not stable by Fourier transformation*.

1.3.2.4.3. *Fourier transforms in* $L^2$

Let $f$ belong to $L^2(\mathbb{R}^n)$, *i.e.* be such that

$$\|f\|_2 = \left(\int_{\mathbb{R}^n} |f(\mathbf{x})|^2 \,\mathrm{d}^n\mathbf{x}\right)^{1/2} < \infty.$$

1.3.2.4.3.1. *Invariance of* $L^2$

$\mathcal{F}[f]$ and $\bar{\mathcal{F}}[f]$ exist and are functions in $L^2$, *i.e.* $\mathcal{F}L^2 = L^2$, $\bar{\mathcal{F}}L^2 = L^2$.

1.3.2.4.3.2. *Reciprocity*

$\mathcal{F}[\bar{\mathcal{F}}[f]] = f$ and $\bar{\mathcal{F}}[\mathcal{F}[f]] = f$, equality being taken as 'almost everywhere' equality. This again leads to calling $\bar{\mathcal{F}}$ the 'inverse Fourier transformation' rather than the Fourier cotransformation.

1.3.2.4.3.3. *Isometry*

$\mathcal{F}$ and $\bar{\mathcal{F}}$ preserve the $L^2$ norm:

$$\|\mathcal{F}[f]\|_2 = \|\bar{\mathcal{F}}[f]\|_2 = \|f\|_2 \text{ (Parseval's/Plancherel's theorem).}$$

This property, which may be written in terms of the inner product (,) in $L^2(\mathbb{R}^n)$ as

$$(\mathcal{F}[f], \mathcal{F}[g]) = (\bar{\mathcal{F}}[f], \bar{\mathcal{F}}[g]) = (f, g),$$

implies that $\mathcal{F}$ and $\bar{\mathcal{F}}$ are *unitary* transformations of $L^2(\mathbb{R}^n)$ into itself, *i.e.* infinite-dimensional 'rotations'.

1.3.2.4.3.4. *Eigenspace decomposition of* $L^2$

Some light can be shed on the geometric structure of these rotations by the following simple considerations. Note that

$$\begin{aligned}\mathcal{F}^2[f](\mathbf{x}) &= \int_{\mathbb{R}^n} \mathcal{F}[f](\boldsymbol{\xi}) \exp(-2\pi i\mathbf{x} \cdot \boldsymbol{\xi}) \,\mathrm{d}^n\boldsymbol{\xi} \\ &= \bar{\mathcal{F}}[\mathcal{F}[f]](-\mathbf{x}) = f(-\mathbf{x})\end{aligned}$$

so that $\mathcal{F}^4$ (and similarly $\bar{\mathcal{F}}^4$) is the identity map. Any eigenvalue of $\mathcal{F}$ or $\bar{\mathcal{F}}$ is therefore a fourth root of unity, *i.e.* $\pm 1$ or $\pm i$, and $L^2(\mathbb{R}^n)$ splits into an orthogonal direct sum

$$\mathbf{H}_0 \otimes \mathbf{H}_1 \otimes \mathbf{H}_2 \otimes \mathbf{H}_3,$$

where $\mathcal{F}$ (respectively $\bar{\mathcal{F}}$) acts in each subspace $\mathbf{H}_k (k = 0, 1, 2, 3)$ by multiplication by $(-i)^k$. Orthonormal bases for these subspaces can be constructed from Hermite functions (*cf.* Section 1.3.2.4.4.2) This method was used by Wiener (1933, pp. 51–71).

1.3.2.4.3.5. *The convolution theorem and the isometry property*

In $L^2$, the convolution theorem (when applicable) and the Parseval/Plancherel theorem are not independent. Suppose that $f$, $g$, $f \times g$ and $f * g$ are all in $L^2$ (without questioning whether these properties are independent). Then $f * g$ may be written in terms of the inner product in $L^2$ as follows:

$$(f * g)(\mathbf{x}) = \int_{\mathbb{R}^n} f(\mathbf{x} - \mathbf{y})g(\mathbf{y}) \,\mathrm{d}^n\mathbf{y} = \int_{\mathbb{R}^n} \overline{\breve{\bar{f}}(\mathbf{y} - \mathbf{x})}g(\mathbf{y}) \,\mathrm{d}^n\mathbf{y},$$

*i.e.*

$$(f * g)(\mathbf{x}) = (\tau_{\mathbf{x}} \breve{\bar{f}}, g).$$

Invoking the isometry property, we may rewrite the right-hand side as





references