*International Tables for Crystallography* (2006). Vol. B, Section 1.3.2.5.4, p. 39.



Besides the general references to distribution theory mentioned in Section 1.3.2.3.1 the reader may consult the books by Zemanian (1965, 1968). Lavoine (1963) contains tables of Fourier transforms of distributions.

#### 1.3.2.5.2. *$\mathscr{S}$ as a test-function space*

A notion of convergence has to be introduced in $\mathscr{S}(\mathbb{R}^n)$ in order to be able to define and test the continuity of linear functionals on it.

A sequence $(\varphi_j)$ of functions in $\mathscr{S}$ will be said to converge to 0 if, for any given multi-indices $\mathbf{k}$ and $\mathbf{p}$, the sequence $(\mathbf{x}^{\mathbf{k}} D^{\mathbf{p}} \varphi_j)$ tends to 0 uniformly on $\mathbb{R}^n$.

It can be shown that $\mathscr{D}(\mathbb{R}^n)$ is dense in $\mathscr{S}(\mathbb{R}^n)$. Translation is continuous for this topology. For any linear differential operator $P(D) = \sum_{\mathbf{p}} a_{\mathbf{p}} D^{\mathbf{p}}$ and any polynomial $Q(\mathbf{x})$ over $\mathbb{R}^n$, $(\varphi_j) \to 0$ implies $[Q(\mathbf{x}) \times P(D)\varphi_j] \to 0$ in the topology of $\mathscr{S}$. Therefore, differentiation and multiplication by polynomials are continuous for the topology on $\mathscr{S}$.

The Fourier transformations $\mathscr{F}$ and $\bar{\mathscr{F}}$ are also continuous for the topology of $\mathscr{S}$. Indeed, let $(\varphi_j)$ converge to 0 for the topology on $\mathscr{S}$. Then, by Section 1.3.2.4.2,

$$\|(2\pi\boldsymbol{\xi})^{\mathbf{m}} D^{\mathbf{p}}(\mathscr{F}[\varphi_j])\|_\infty \leq \|D^{\mathbf{m}}[(2\pi\mathbf{x})^{\mathbf{p}} \varphi_j]\|_1.$$

The right-hand side tends to 0 as $j \to \infty$ by definition of convergence in $\mathscr{S}$, hence $\|\boldsymbol{\xi}\|^{\mathbf{m}} D^{\mathbf{p}}(\mathscr{F}[\varphi_j]) \to 0$ uniformly, so that $(\mathscr{F}[\varphi_j]) \to 0$ in $\mathscr{S}$ as $j \to \infty$. The same proof applies to $\bar{\mathscr{F}}$.

#### 1.3.2.5.3. *Definition and examples of tempered distributions*

A distribution $T \in \mathscr{D}'(\mathbb{R}^n)$ is said to be *tempered* if it can be extended into a continuous linear functional on $\mathscr{S}$.

If $\mathscr{S}'(\mathbb{R}^n)$ is the topological dual of $\mathscr{S}(\mathbb{R}^n)$, and if $S \in \mathscr{S}'(\mathbb{R}^n)$, then its restriction to $\mathscr{D}$ is a tempered distribution; conversely, if $T \in \mathscr{D}'$ is tempered, then its extension to $\mathscr{S}$ is unique (because $\mathscr{D}$ is dense in $\mathscr{S}$), hence it defines an element $S$ of $\mathscr{S}'$. We may therefore identify $\mathscr{S}'$ and the space of tempered distributions.

A distribution with compact support is tempered, *i.e.* $\mathscr{S}' \supset \mathscr{E}'$. By transposition of the corresponding properties of $\mathscr{S}$, it is readily established that the derivative, translate or product by a polynomial of a tempered distribution is still a tempered distribution.

These inclusion relations may be summarized as follows: since $\mathscr{S}$ contains $\mathscr{D}$ but is contained in $\mathscr{E}$, the reverse inclusions hold for the topological duals, and hence $\mathscr{S}'$ contains $\mathscr{E}'$ but is contained in $\mathscr{D}'$.

A locally summable function $f$ on $\mathbb{R}^n$ will be said to be *of polynomial growth* if $|f(\mathbf{x})|$ can be majorized by a polynomial in $\|\mathbf{x}\|$ as $\|\mathbf{x}\| \to \infty$. It is easily shown that such a function $f$ defines a tempered distribution $T_f$ *via*

$$\langle T_f, \varphi \rangle = \int_{\mathbb{R}^n} f(\mathbf{x})\varphi(\mathbf{x})\, \mathrm{d}^n\mathbf{x}.$$

In particular, polynomials over $\mathbb{R}^n$ define tempered distributions, and so do functions in $\mathscr{S}$. The latter remark, together with the transposition identity (Section 1.3.2.4.4), invites the extension of $\mathscr{F}$ and $\bar{\mathscr{F}}$ from $\mathscr{S}$ to $\mathscr{S}'$.

#### 1.3.2.5.4. *Fourier transforms of tempered distributions*

The Fourier transform $\mathscr{F}[T]$ and cotransform $\bar{\mathscr{F}}[T]$ of a tempered distribution $T$ are defined by

$$\langle \mathscr{F}[T], \varphi \rangle = \langle T, \mathscr{F}[\varphi] \rangle$$
$$\langle \bar{\mathscr{F}}[T], \varphi \rangle = \langle T, \bar{\mathscr{F}}[\varphi] \rangle$$

for all test functions $\varphi \in \mathscr{S}$. Both $\mathscr{F}[T]$ and $\bar{\mathscr{F}}[T]$ are themselves tempered distributions, since the maps $\varphi \longmapsto \mathscr{F}[\varphi]$ and $\varphi \longmapsto \bar{\mathscr{F}}[\varphi]$ are both linear and continuous for the topology of $\mathscr{S}$. In the same way that $\mathbf{x}$ and $\boldsymbol{\xi}$ have been used consistently as arguments for $\varphi$ and $\mathscr{F}[\varphi]$, respectively, the notation $T_{\mathbf{x}}$ and $\mathscr{F}[T]_{\boldsymbol{\xi}}$ will be used to indicate which variables are involved.

When $T$ is a distribution with compact support, its Fourier transform may be written

$$\mathscr{F}[T_{\mathbf{x}}]_{\boldsymbol{\xi}} = \langle T_{\mathbf{x}}, \exp(-2\pi i \boldsymbol{\xi} \cdot \mathbf{x}) \rangle$$

since the function $\mathbf{x} \longmapsto \exp(-2\pi i \boldsymbol{\xi} \cdot \mathbf{x})$ is in $\mathscr{E}$ while $T_{\mathbf{x}} \in \mathscr{E}'$. It can be shown, as in Section 1.3.2.4.2, to be analytically continuable into an entire function over $\mathbb{C}^n$.

#### 1.3.2.5.5. *Transposition of basic properties*

The duality between differentiation and multiplication by a monomial extends from $\mathscr{S}$ to $\mathscr{S}'$ by transposition:

$$\mathscr{F}[D_{\mathbf{x}}^{\mathbf{p}} T_{\mathbf{x}}]_{\boldsymbol{\xi}} = (2\pi i \boldsymbol{\xi})^{\mathbf{p}} \mathscr{F}[T_{\mathbf{x}}]_{\boldsymbol{\xi}}$$
$$D_{\boldsymbol{\xi}}^{\mathbf{p}}(\mathscr{F}[T_{\mathbf{x}}]_{\boldsymbol{\xi}}) = \mathscr{F}[(-2\pi i \mathbf{x})^{\mathbf{p}} T_{\mathbf{x}}]_{\boldsymbol{\xi}}.$$

Analogous formulae hold for $\bar{\mathscr{F}}$, with $i$ replaced by $-i$.

The formulae expressing the duality between translation and phase shift, *e.g.*

$$\mathscr{F}[\tau_{\mathbf{a}} T_{\mathbf{x}}]_{\boldsymbol{\xi}} = \exp(-2\pi i \mathbf{a} \cdot \boldsymbol{\xi}) \mathscr{F}[T_{\mathbf{x}}]_{\boldsymbol{\xi}}$$
$$\tau_{\boldsymbol{\alpha}}(\mathscr{F}[T_{\mathbf{x}}]_{\boldsymbol{\xi}}) = \mathscr{F}[\exp(2\pi i \boldsymbol{\alpha} \cdot \mathbf{x}) T_{\mathbf{x}}]_{\boldsymbol{\xi}};$$

between a linear change of variable and its contragredient, *e.g.*

$$\mathscr{F}[A^{\#} T] = |\det \mathbf{A}|[(\mathbf{A}^{-1})^T]^{\#} \mathscr{F}[T];$$

are obtained similarly by transposition from the corresponding identities in $\mathscr{S}$. They give a transposition formula for an affine change of variables $\mathbf{x} \longmapsto S(\mathbf{x}) = \mathbf{A}\mathbf{x} + \mathbf{b}$ with non-singular matrix $\mathbf{A}$:

$$\begin{aligned} \mathscr{F}[S^{\#} T] &= \exp(-2\pi i \boldsymbol{\xi} \cdot \mathbf{b}) \mathscr{F}[A^{\#} T] \\ &= \exp(-2\pi i \boldsymbol{\xi} \cdot \mathbf{b}) |\det \mathbf{A}|[(\mathbf{A}^{-1})^T]^{\#} \mathscr{F}[T], \end{aligned}$$

with a similar result for $\bar{\mathscr{F}}$, replacing $-i$ by $+i$.

Conjugate symmetry is obtained similarly:

$$\mathscr{F}[\bar{T}] = \overline{\mathscr{F}[T]}^{\vee}, \mathscr{F}[\breve{\bar{T}}] = \overline{\mathscr{F}[T]},$$

with the same identities for $\bar{\mathscr{F}}$.

The tensor product property also transposes to tempered distributions: if $U \in \mathscr{S}'(\mathbb{R}^m)$, $V \in \mathscr{S}'(\mathbb{R}^n)$,

$$\mathscr{F}[U_{\mathbf{x}} \otimes V_{\mathbf{y}}] = \mathscr{F}[U]_{\boldsymbol{\xi}} \otimes \mathscr{F}[V]_{\boldsymbol{\eta}}$$
$$\bar{\mathscr{F}}[U_{\mathbf{x}} \otimes V_{\mathbf{y}}] = \bar{\mathscr{F}}[U]_{\boldsymbol{\xi}} \otimes \bar{\mathscr{F}}[V]_{\boldsymbol{\eta}}.$$

#### 1.3.2.5.6. *Transforms of $\delta$-functions*

Since $\delta$ has compact support,

$$\mathscr{F}[\delta_{\mathbf{x}}]_{\boldsymbol{\xi}} = \langle \delta_{\mathbf{x}}, \exp(-2\pi i \boldsymbol{\xi} \cdot \mathbf{x}) \rangle = 1_{\boldsymbol{\xi}}, \quad \textit{i.e. } \mathscr{F}[\delta] = 1.$$

It is instructive to show that conversely $\mathscr{F}[1] = \delta$ without invoking the reciprocity theorem. Since $\partial_j 1 = 0$ for all $j = 1, \ldots, n$, it follows from Section 1.3.2.3.9.4 that $\mathscr{F}[1] = c\delta$; the constant $c$ can be determined by using the invariance of the standard Gaussian $G$ established in Section 1.3.2.4.3:

$$\langle \mathscr{F}[1]_{\mathbf{x}}, G_{\mathbf{x}} \rangle = \langle 1_{\boldsymbol{\xi}}, G_{\boldsymbol{\xi}} \rangle = 1;$$

hence $c = 1$. Thus, $\mathscr{F}[1] = \delta$.

The basic properties above then read (using multi-indices to denote differentiation):