$$\text{(i)} \qquad w_{\boldsymbol{\mu}} = \langle T^0_{\mathbf{x}}, \exp(-2\pi i \boldsymbol{\mu}\cdot\mathbf{x})\rangle$$

$$\text{(ii)} \qquad T_{\mathbf{x}} = \sum_{\boldsymbol{\mu}\in\mathbb{Z}^n} w_{\boldsymbol{\mu}} \exp(+2\pi i \boldsymbol{\mu}\cdot\mathbf{x})$$

are referred to as the *Fourier analysis* and the *Fourier synthesis* of T, respectively (there is a discrepancy between this terminology and the crystallographic one, see Section 1.3.4.2.1.1). In other words, any periodic distribution $T \in \mathscr{S}'$ may be represented by a Fourier series (ii), whose coefficients are calculated by (i). The convergence of (ii) towards T in $\mathscr{S}'$ will be investigated later (Section 1.3.2.6.10).

1.3.2.6.5. *The case of non-standard period lattices*

Let Λ denote the non-standard lattice consisting of all vectors of the form $\sum_{j=1} m_j \mathbf{a}_j$, where the m_j are rational integers and $\mathbf{a}_1, \ldots, \mathbf{a}_n$ are n linearly independent vectors in $\mathbb{R}^n$. Let R be the corresponding lattice distribution: $R = \sum_{x\in\Lambda} \delta_{(\mathbf{x})}$.

Let $\mathbf{A}$ be the non-singular $n\times n$ matrix whose successive columns are the coordinates of vectors $\mathbf{a}_1, \ldots, \mathbf{a}_n$ in the standard basis of $\mathbb{R}^n$; $\mathbf{A}$ will be called the *period matrix* of Λ, and the mapping $\mathbf{x} \longmapsto \mathbf{Ax}$ will be denoted by A. According to Section 1.3.2.3.9.5 we have

$$\langle R, \varphi\rangle = \sum_{\mathbf{m}\in\mathbb{Z}^n} \varphi(\mathbf{Am}) = \langle r, (A^{-1})^{\#}\varphi\rangle = |\det \mathbf{A}|^{-1}\langle A^{\#} r, \varphi\rangle$$

for any $\varphi \in \mathscr{S}$, and hence $R = |\det \mathbf{A}|^{-1} A^{\#} r$. By Fourier transformation, according to Section 1.3.2.5.5,

$$\mathscr{F}[R] = |\det \mathbf{A}|^{-1} \mathscr{F}[A^{\#} r] = [(\mathbf{A}^{-1})^T]^{\#} \mathscr{F}[r] = [(\mathbf{A}^{-1})^T]^{\#} r,$$

which we write:

$$\mathscr{F}[R] = |\det \mathbf{A}|^{-1} R^*$$

with

$$R^* = |\det \mathbf{A}| [(\mathbf{A}^{-1})^T]^{\#} r.$$

R^* is a lattice distribution:

$$R^* = \sum_{\boldsymbol{\mu}\in\mathbb{Z}^n} \delta_{[(\mathbf{A}^{-1})^T \boldsymbol{\mu}]} = \sum_{\boldsymbol{\xi}\in\Lambda^*} \delta_{(\boldsymbol{\xi})}$$

associated with the *reciprocal lattice* Λ^* whose basis vectors $\mathbf{a}^*_1, \ldots, \mathbf{a}^*_n$ are the columns of $(\mathbf{A}^{-1})^T$. Since the latter matrix is equal to the adjoint matrix (*i.e.* the matrix of co-factors) of $\mathbf{A}$ divided by $\det \mathbf{A}$, the components of the reciprocal basis vectors can be written down explicitly (see Section 1.3.4.2.1.1 for the crystallographic case $n = 3$).

A distribution T will be called Λ-*periodic* if $\tau_{\boldsymbol{\xi}} T = T$ for all $\boldsymbol{\xi} \in \Lambda$; as previously, T may be written $R * T^0$ for some motif distribution T^0 with compact support. By Fourier transformation,

$$\begin{aligned}\mathscr{F}[T] &= |\det \mathbf{A}|^{-1} R^* \cdot \mathscr{F}[T^0]\\ &= |\det \mathbf{A}|^{-1} \sum_{\boldsymbol{\xi}\in\Lambda^*} \mathscr{F}[T^0](\boldsymbol{\xi})\delta_{(\boldsymbol{\xi})}\\ &= |\det \mathbf{A}|^{-1} \sum_{\boldsymbol{\mu}\in\mathbb{Z}^n} \mathscr{F}[T^0][(\mathbf{A}^{-1})^T\boldsymbol{\mu}]\delta_{[(\mathbf{A}^{-1})^T\boldsymbol{\mu}]}\end{aligned}$$

so that $\mathscr{F}[T]$ is a weighted reciprocal-lattice distribution, the weight attached to node $\boldsymbol{\xi} \in \Lambda^*$ being $|\det \mathbf{A}|^{-1}$ times the value $\mathscr{F}[T^0](\boldsymbol{\xi})$ of the Fourier transform of the motif T^0.

This result may be further simplified if T and its motif T^0 are referred to the standard period lattice $\mathbb{Z}^n$ by defining t and t^0 so that $T = A^{\#} t$, $T^0 = A^{\#} t^0$, $t = r * t^0$. Then

$$\mathscr{F}[T^0](\boldsymbol{\xi}) = |\det \mathbf{A}| \mathscr{F}[t^0](\mathbf{A}^T\boldsymbol{\xi}),$$

hence

$$\mathscr{F}[T^0][(\mathbf{A}^{-1})^T\boldsymbol{\mu}] = |\det \mathbf{A}| \mathscr{F}[t^0](\boldsymbol{\mu}),$$

so that

$$\mathscr{F}[T] = \sum_{\boldsymbol{\mu}\in\mathbb{Z}^n} \mathscr{F}[t^0](\boldsymbol{\mu})\delta_{[(\mathbf{A}^{-1})^T\boldsymbol{\mu}]}$$

in non-standard coordinates, while

$$\mathscr{F}[t] = \sum_{\boldsymbol{\mu}\in\mathbb{Z}^n} \mathscr{F}[t^0](\boldsymbol{\mu})\delta_{(\boldsymbol{\mu})}$$

in standard coordinates.

The reciprocity theorem may then be written:

$$\text{(iii)} \qquad W_{\boldsymbol{\xi}} = |\det \mathbf{A}|^{-1}\langle T^0_{\mathbf{x}}, \exp(-2\pi i \boldsymbol{\xi}\cdot\mathbf{x})\rangle, \quad \boldsymbol{\xi} \in \boldsymbol{\Lambda}^*$$

$$\text{(iv)} \qquad T_{\mathbf{x}} = \sum_{\boldsymbol{\xi}\in\Lambda^*} W_{\boldsymbol{\xi}} \exp(+2\pi i \boldsymbol{\xi}\cdot\mathbf{x})$$

in non-standard coordinates, or equivalently:

$$\text{(v)} \qquad w_{\boldsymbol{\mu}} = \langle t^0_{\mathbf{x}}, \exp(-2\pi i \boldsymbol{\mu}\cdot\mathbf{x})\rangle, \quad \boldsymbol{\mu} \in \mathbb{Z}^n$$

$$\text{(vi)} \qquad t_{\mathbf{x}} = \sum_{\boldsymbol{\mu}\in\mathbb{Z}^n} w_{\boldsymbol{\mu}} \exp(+2\pi i \boldsymbol{\mu}\cdot\mathbf{x})$$

in standard coordinates. It gives an n-dimensional Fourier series representation for any periodic distribution over $\mathbb{R}^n$. The convergence of such series in $\mathscr{S}'(\mathbb{R}^n)$ will be examined in Section 1.3.2.6.10.

1.3.2.6.6. *Duality between periodization and sampling*

Let T^0 be a distribution with compact support (the 'motif'). Its Fourier transform $\bar{\mathscr{F}}[T^0]$ is analytic (Section 1.3.2.5.4) and may thus be used as a multiplier.

We may rephrase the preceding results as follows:

(i) if T^0 is 'periodized by R' to give $R * T^0$, then $\bar{\mathscr{F}}[T^0]$ is 'sampled by R^*' to give $|\det \mathbf{A}|^{-1} R^* \cdot \bar{\mathscr{F}}[T^0]$;

(ii) if $\bar{\mathscr{F}}[T^0]$ is 'sampled by R^*' to give $R^* \cdot \bar{\mathscr{F}}[T^0]$, then T^0 is 'periodized by R' to give $|\det \mathbf{A}| R * T^0$.

Thus the Fourier transformation establishes a duality between the periodization of a distribution by a period lattice Λ and the sampling of its transform at the nodes of lattice Λ^* reciprocal to Λ. This is a particular instance of the convolution theorem of Section 1.3.2.5.8.

At this point it is traditional to break the symmetry between $\mathscr{F}$ and $\bar{\mathscr{F}}$ which distribution theory has enabled us to preserve even in the presence of periodicity, and to perform two distinct identifications:

(i) a Λ-periodic distribution T will be handled as a distribution $\tilde{T}$ on $\mathbb{R}^n/\Lambda$, was done in Section 1.3.2.6.3;

(ii) a weighted lattice distribution $W = \sum_{\boldsymbol{\mu}\in\mathbb{Z}^n} W_{\boldsymbol{\mu}}\delta_{[(\mathbf{A}^{-1})^T\boldsymbol{\mu}]}$ will be identified with the collection $\{W_{\boldsymbol{\mu}} | \boldsymbol{\mu} \in \mathbb{Z}^n\}$ of its n-tuply indexed coefficients.

1.3.2.6.7. *The Poisson summation formula*

Let $\varphi \in \mathscr{S}$, so that $\mathscr{F}[\varphi] \in \mathscr{S}$. Let R be the lattice distribution associated to lattice Λ, with period matrix $\mathbf{A}$, and let R^* be associated to the reciprocal lattice Λ^*. Then we may write:

$$\begin{aligned}\langle R, \varphi\rangle &= \langle R, \bar{\mathscr{F}}[\mathscr{F}[\varphi]]\rangle\\ &= \langle \bar{\mathscr{F}}[R], \mathscr{F}[\varphi]\rangle\\ &= |\det \mathbf{A}|^{-1}\langle R^*, \mathscr{F}[\varphi]\rangle\end{aligned}$$

i.e.

references