$S_{\mathbf{B}/\mathbf{A}}$ (respectively $S^*_{\mathbf{A}/\mathbf{B}}$) on the one hand, and *period decimation* by 'dilation' by $\mathbf{D}^{\#}$ on the other hand, are mutually inverse operations on $R_{\mathbf{A}}$ and $R_{\mathbf{B}}$ (respectively $R^*_{\mathbf{A}}$ and $R^*_{\mathbf{B}}$).

1.3.2.7.2.4. *Relation between Fourier transforms*

Finally, let us consider the relations between the *Fourier transforms* of these lattice distributions. Recalling the basic relation of Section 1.3.2.6.5,

$$\begin{aligned}\mathscr{F}[R_{\mathbf{A}}] &= \frac{1}{|\det \mathbf{A}|} R^*_{\mathbf{A}} \\ &= \frac{1}{|\det \mathbf{DB}|} T^*_{\mathbf{A}/\mathbf{B}} * R^*_{\mathbf{B}} \qquad \text{by (ii)}^* \\ &= \left(\frac{1}{|\det \mathbf{D}|} T^*_{\mathbf{A}/\mathbf{B}}\right) * \left(\frac{1}{|\det \mathbf{B}|} R^*_{\mathbf{B}}\right)\end{aligned}$$

i.e.

$$\text{(iv)} \qquad \mathscr{F}[R_{\mathbf{A}}] = S^*_{\mathbf{A}/\mathbf{B}} * \mathscr{F}[R_{\mathbf{B}}]$$

and similarly:

$$\text{(v)} \qquad \mathscr{F}[R^*_{\mathbf{B}}] = S_{\mathbf{B}/\mathbf{A}} * \mathscr{F}[R^*_{\mathbf{A}}].$$

Thus $R_{\mathbf{A}}$ (respectively $R^*_{\mathbf{B}}$), a *decimated* version of $R_{\mathbf{B}}$ (respectively $R^*_{\mathbf{A}}$), is transformed by $\mathscr{F}$ into a *subdivided* version of $\mathscr{F}[R_{\mathbf{B}}]$ (respectively $\mathscr{F}[R^*_{\mathbf{A}}]$).

The converse is also true:

$$\begin{aligned}\mathscr{F}[R_{\mathbf{B}}] &= \frac{1}{|\det \mathbf{B}|} R^*_{\mathbf{B}} \\ &= \frac{1}{|\det \mathbf{B}|}\frac{1}{|\det \mathbf{D}|} (\mathbf{D}^T)^{\#} R^*_{\mathbf{A}} \qquad \text{by (i)}^* \\ &= (\mathbf{D}^T)^{\#}\left(\frac{1}{|\det \mathbf{A}|} R^*_{\mathbf{A}}\right)\end{aligned}$$

i.e.

$$\text{(iv}'\text{)} \qquad \mathscr{F}[R_{\mathbf{B}}] = (\mathbf{D}^T)^{\#} \mathscr{F}[R_{\mathbf{A}}]$$

and similarly

$$\text{(v}'\text{)} \qquad \mathscr{F}[R^*_{\mathbf{A}}] = \mathbf{D}^{\#} \mathscr{F}[R^*_{\mathbf{B}}].$$

Thus $R_{\mathbf{B}}$ (respectively $R^*_{\mathbf{A}}$), a *subdivided* version of $R_{\mathbf{A}}$ (respectively $R^*_{\mathbf{B}}$) is transformed by $\mathscr{F}$ into a *decimated* version of $\mathscr{F}[R_{\mathbf{A}}]$ (respectively $\mathscr{F}[R^*_{\mathbf{B}}]$). Therefore, *the Fourier transform exchanges subdivision and decimation of period lattices for lattice distributions*.

Further insight into this phenomenon is provided by applying $\bar{\mathscr{F}}$ to both sides of (iv) and (v) and invoking the convolution theorem:

$$\text{(iv}''\text{)} \qquad R_{\mathbf{A}} = \bar{\mathscr{F}}[S^*_{\mathbf{A}/\mathbf{B}}] \times R_{\mathbf{B}}$$

$$\text{(v}''\text{)} \qquad R^*_{\mathbf{B}} = \bar{\mathscr{F}}[S_{\mathbf{B}/\mathbf{A}}] \times R^*_{\mathbf{A}}.$$

These identities show that multiplication by the transform of the period-subdividing distribution $S^*_{\mathbf{A}/\mathbf{B}}$ (respectively $S_{\mathbf{B}/\mathbf{A}}$) has the effect of decimating $R_{\mathbf{B}}$ to $R_{\mathbf{A}}$ (respectively $R^*_{\mathbf{A}}$ to $R^*_{\mathbf{B}}$). They clearly imply that, if $\ell \in \Lambda_{\mathbf{B}}/\Lambda_{\mathbf{A}}$ and $\ell^* \in \Lambda^*_{\mathbf{A}}/\Lambda^*_{\mathbf{B}}$, then

$$\begin{aligned}\bar{\mathscr{F}}[S^*_{\mathbf{A}/\mathbf{B}}](\ell) &= 1 \text{ if } \ell = \mathbf{0} \qquad (\textit{i.e.}\text{ if } \ell \text{ belongs to the class of } \Lambda_{\mathbf{A}}), \\ &= 0 \text{ if } \ell \neq \mathbf{0}; \\ \bar{\mathscr{F}}[S_{\mathbf{B}/\mathbf{A}}](\ell^*) &= 1 \text{ if } \ell^* = \mathbf{0} \qquad (\textit{i.e.}\text{ if } \ell^* \text{ belongs to the class of } \Lambda^*_{\mathbf{B}}), \\ &= 0 \text{ if } \ell^* \neq \mathbf{0}.\end{aligned}$$

Therefore, the duality between subdivision and decimation may be viewed as another aspect of that between convolution and multiplication.

There is clearly a strong analogy between the sampling/periodization duality of Section 1.3.2.6.6 and the decimation/subdivision duality, which is viewed most naturally in terms of subgroup relationships: both sampling and decimation involve restricting a function to a *discrete additive subgroup* of the domain over which it is initially given.

1.3.2.7.2.5. *Sublattice relations in terms of periodic distributions*

The usual presentation of this duality is not in terms of lattice distributions, but of periodic distributions obtained by convolving them with a motif.

Given $T^0 \in \mathscr{E}'(\mathbb{R}^n)$, let us form $R_{\mathbf{A}} * T^0$, then *decimate* its transform $(1/|\det \mathbf{A}|)R^*_{\mathbf{A}} \times \bar{\mathscr{F}}[T^0]$ by keeping only its values at the points of the coarser lattice $\Lambda^*_{\mathbf{B}} = \mathbf{D}^T \Lambda^*_{\mathbf{A}}$; as a result, $R^*_{\mathbf{A}}$ is replaced by $(1/|\det \mathbf{D}|)R^*_{\mathbf{B}}$, and the reverse transform then yields

$$\frac{1}{|\det \mathbf{D}|} R_{\mathbf{B}} * T^0 = S_{\mathbf{B}/\mathbf{A}} * (R_{\mathbf{A}} * T^0) \qquad \text{by (ii)},$$

which is the *coset-averaged* version of the original $R_{\mathbf{A}} * T^0$. The converse situation is analogous to that of Shannon's sampling theorem. Let a function $\varphi \in \mathscr{E}(\mathbb{R}^n)$ whose transform $\Phi = \mathscr{F}[\varphi]$ has compact support be sampled as $R_{\mathbf{B}} \times \varphi$ at the nodes of $\Lambda_{\mathbf{B}}$. Then

$$\mathscr{F}[R_{\mathbf{B}} \times \varphi] = \frac{1}{|\det \mathbf{B}|}(R^*_{\mathbf{B}} * \Phi)$$

is periodic with period lattice $\Lambda^*_{\mathbf{B}}$. If the sampling lattice $\Lambda_{\mathbf{B}}$ is decimated to $\Lambda_{\mathbf{A}} = \mathbf{D}\Lambda_{\mathbf{B}}$, the inverse transform becomes

$$\begin{aligned}\mathscr{F}[R_{\mathbf{A}} \times \varphi] &= \frac{1}{|\det \mathbf{D}|}(R^*_{\mathbf{A}} * \Phi) \\ &= S^*_{\mathbf{A}/\mathbf{B}} * (R^*_{\mathbf{B}} * \Phi) \qquad \text{by (ii)}^*,\end{aligned}$$

hence becomes periodized more finely by averaging over the cosets of $\Lambda^*_{\mathbf{A}}/\Lambda^*_{\mathbf{B}}$. With this finer periodization, the various copies of Supp Φ may start to overlap (a phenomenon called 'aliasing'), indicating that decimation has produced too coarse a sampling of φ.

1.3.2.7.3. *Discretization of the Fourier transformation*

Let $\varphi^0 \in \mathscr{E}(\mathbb{R}^n)$ be such that $\Phi^0 = \mathscr{F}[\varphi^0]$ has compact support (φ^0 is said to be *band-limited*). Then $\varphi = R_{\mathbf{A}} * \varphi^0$ is $\Lambda_{\mathbf{A}}$-periodic, and $\Phi = \mathscr{F}[\varphi] = (1/|\det \mathbf{A}|)R^*_{\mathbf{A}} \times \Phi^0$ is such that only a finite number of points $\lambda^*_{\mathbf{A}}$ of $\Lambda^*_{\mathbf{A}}$ have a non-zero Fourier coefficient $\Phi^0(\lambda^*_{\mathbf{A}})$ attached to them. We may therefore find a *decimation* $\Lambda^*_{\mathbf{B}} = \mathbf{D}^T \Lambda^*_{\mathbf{A}}$ of $\Lambda^*_{\mathbf{A}}$ such that the distinct translates of Supp Φ^0 by vectors of $\Lambda^*_{\mathbf{B}}$ do not intersect.

The distribution Φ can be uniquely recovered from $R^*_{\mathbf{B}} * \Phi$ by the procedure of Section 1.3.2.7.1, and we may write:

$$\begin{aligned}R^*_{\mathbf{B}} * \Phi &= \frac{1}{|\det \mathbf{A}|} R^*_{\mathbf{B}} * (R^*_{\mathbf{A}} \times \Phi^0) \\ &= \frac{1}{|\det \mathbf{A}|} R^*_{\mathbf{A}} \times (R^*_{\mathbf{B}} * \Phi^0) \\ &= \frac{1}{|\det \mathbf{A}|} R^*_{\mathbf{B}} * [T^*_{\mathbf{A}/\mathbf{B}} \times (R^*_{\mathbf{B}} * \Phi^0)];\end{aligned}$$

these rearrangements being legitimate because Φ^0 and $T^*_{\mathbf{A}/\mathbf{B}}$ have compact supports which are intersection-free under the action of $\Lambda^*_{\mathbf{B}}$. By virtue of its $\Lambda^*_{\mathbf{B}}$-periodicity, this distribution is entirely characterized by its 'motif' $\tilde{\Phi}$ with respect to $\Lambda^*_{\mathbf{B}}$: