

## 2.1. STATISTICAL PROPERTIES OF THE WEIGHTED RECIPROCAL LATTICE

denominators, but not in the numerators, of equation (2.1.2.6) by their arguments, and this is equivalent to the approximation  $\sin x \approx x$  in the denominators only. This is a good approximation for atoms close together in the structure and thus giving the largest terms in the sums in equation (2.1.2.6), and gives the correct sign and order of magnitude even for  $x$  having its maximum value of  $\pi/2$ .

## 2.1.2.2. Physical background

The preceding section has used mathematical arguments. From a physical point of view, the radiation diffracted by atoms that are resolved will interfere destructively, so that the resulting intensity will be the sum of the intensities diffracted by individual atoms, whereas that from completely unresolved atoms will interfere constructively, so that *amplitudes* rather than intensities add. In intermediate cases there will be partial constructive interference. Resolution in accordance with the Rayleigh (1879) criterion requires that  $s = (2 \sin \theta) / \lambda$  should be greater than half the reciprocal of the minimum interatomic distance in the crystal (Wilson, 1979); full resolution requires a substantial multiple of this. This criterion is essentially equivalent to that proposed from the study of a special case of the second process in the preceding section.

## 2.1.2.3. An approximation for organic compounds

In organic compounds there are very many interatomic distances of about 1.5 or 1.4 Å. Adoption of the preceding criterion would mean that the inner portion of the region of reciprocal space accessible by the use of copper  $K\alpha$  radiation is not within the sphere of intensity statistics based on fixed-index (first process) averaging. No substantial results are available for fixed-parameter (second process) averaging, and very few from the approximation to it (third process).

To the extent to which the third process is acceptable, an approximation to the variation of  $\langle I \rangle$  with  $\sin \theta$  is obtainable. The exponent in equation (2.1.2.2) can be written as

$$2\pi i s r_{jk} \cos \psi, \quad (2.1.2.9)$$

where  $s$  is the radial distance in reciprocal space,  $r_{jk}$  is the distance from the  $j$ th to the  $k$ th atom and  $\psi$  is the angle between the vectors  $s$  and  $r$ . Averaging over a sphere of radius  $s$ , with  $\psi$  treated as the co-latitude, gives

$$\langle I \rangle = \sum_j \sum_k f_j f_k \frac{\sin 2\pi s r_{jk}}{2\pi s r_{jk}}. \quad (2.1.2.10)$$

This is the familiar Debye expression. It has the correct limits for  $s$  zero and  $s$  large, and is in accord with the argument from resolution.

## 2.1.2.4. Effect of centring

In the preceding discussion there has been a tacit assumption that the lattice is primitive. A centred crystal can always be referred to a primitive lattice and if this is done no change is required. If the centred lattice is retained, many reflections are identically zero and the intensity of the non-zero reflections is enhanced by a factor of two ( $I$  and  $C$  lattices) or four ( $F$  lattice), so that the average intensity of all the reflections, zero and non-zero taken together, is unchanged.

Other symmetry elements affect only zones and rows of reflections, and so do not affect the general average when the total number of reflections is large. Their effect on zones and rows is discussed in Section 2.1.3.

Table 2.1.3.1. Intensity-distribution effects of symmetry elements causing systematic absences

Abbreviations and orientation of axes:  $A$  = acentric distribution,  $C$  = centric distribution,  $Z$  = systematically zero,  $S$  = distribution parameter,  $\langle I \rangle$  = average intensity. Axes are parallel to  $\mathbf{c}$ , planes are perpendicular to  $\mathbf{c}$ .

Element	Reflections	Distribution	$S/\Sigma$	$\langle I \rangle/\Sigma$
$2_1$	$hkl$	$A$	1	1
	$hk0$	$C$	1	1
	$00l$	$(Z + A)/2$	1	2
$3_1, 3_2$	$hkl$	$A$	1	1
	$hk0$	$A$	1	1
	$00l$	$(2Z + A)/3$	1	3
$4_1, 4_3$	$hkl$	$A$	1	1
	$hk0$	$C$	1	1
	$00l$	$(3Z + A)/4$	1	4
$4_2$	$hkl$	$A$	1	1
	$hk0$	$C$	1	1
	$00l$	$(Z + A)/2$	2	4
$6_1, 6_5$	$hkl$	$A$	1	1
	$hk0$	$C$	1	1
	$00l$	$(5Z + A)/6$	1	6
$6_2, 6_4$	$hkl$	$A$	1	1
	$hk0$	$C$	1	1
	$00l$	$(2Z + A)/3$	2	6
$6_3$	$hkl$	$A$	1	1
	$hk0$	$C$	1	1
	$00l$	$(Z + A)/2$	3	6
$a$	$hkl$	$A$	1	1
	$hk0$	$(Z + A)/2$	1	2
	$00l$	$C$	1	1
	$0k0$	$A$	2	2
$C, I$	All	$(Z + A)/2$	1	2
$F$	All	$(3Z + A)/2$	1	4

## 2.1.3. The average intensity of zones and rows

## 2.1.3.1. Symmetry elements producing systematic absences

Symmetry elements can be divided into two types: those that cause systematic absences and those that do not. Those producing systematic absences (glide planes and screw axes) produce at the same time groups of reflections (confined to zones and rows in reciprocal space, respectively) with an average intensity an integral\* multiple of the general average. The effects for single symmetry elements of this type are given in Table 2.1.3.1 for the general reflections  $hkl$  and separately for any zones and rows that are affected. The 'average multipliers' are given in the column headed  $\langle I \rangle/\Sigma$ ; 'distribution' and 'distribution parameters' are treated in Section 2.1.5. As for the centring, the fraction of reflections missing and the integer multiplying the average are related in such a way that the overall intensity is unchanged. The

\* The multiple is given as an exact integer for fixed-index averaging, an approximate integer for fixed-parameter averaging. Statements should be understood to refer to fixed-index averaging unless the contrary is explicitly stated.

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Table 2.1.3.2. *Intensity-distribution effects of symmetry elements not causing systematic absences*

Abbreviations and orientation of axes: *A* = acentric distribution, *C* = centric distribution, *S* = distribution parameter,  $\langle I \rangle$  = average intensity. Axes are parallel to *c*, planes are perpendicular to *c*.

Element	Reflections	Distribution	$S/\Sigma = \langle I \rangle/\Sigma$
1	All	<i>A</i>	1
$\bar{1}$	All	<i>C</i>	1
2	<i>hkl</i>	<i>A</i>	1
	<i>hk0</i>	<i>C</i>	1
	<i>00l</i>	<i>A</i>	2
$\bar{2} = m$	<i>hkl</i>	<i>A</i>	1
	<i>hk0</i>	<i>A</i>	2
	<i>00l</i>	<i>C</i>	1
3	<i>hkl</i>	<i>A</i>	1
	<i>hk0</i>	<i>A</i>	1
	<i>00l</i>	<i>A</i>	3
$\bar{3}$	<i>hkl</i>	<i>C</i>	1
	<i>hk0</i>	<i>C</i>	1
	<i>00l</i>	<i>C</i>	3
4	<i>hkl</i>	<i>A</i>	1
	<i>hk0</i>	<i>C</i>	1
	<i>00l</i>	<i>A</i>	4
$\bar{4}$	<i>hkl</i>	<i>A</i>	1
	<i>hk0</i>	<i>C</i>	1
	<i>00l</i>	<i>C</i>	2
6	<i>hkl</i>	<i>A</i>	1
	<i>hk0</i>	<i>C</i>	1
	<i>00l</i>	<i>A</i>	6
$\bar{6} = 3/m$	<i>hkl</i>	<i>A</i>	1
	<i>hk0</i>	<i>A</i>	2
	<i>00l</i>	<i>C</i>	3

mechanism for compensation for the reflections with enhanced intensity is obvious.

### 2.1.3.2. *Symmetry elements not producing systematic absences*

Certain symmetry elements not producing absences (mirror planes and rotation axes) cause equivalent atoms to coincide in a plane or a line projection and hence produce a zone or row in reciprocal space for which the average intensity is an integral multiple of the general average (Wilson, 1950); the effects of single such symmetry elements are given in Table 2.1.3.2. There is, however, no obvious mechanism for compensation for this enhancement. When reflections are few this may be an important matter in assigning an approximate absolute scale by comparing observed and calculated intensities. Wilson (1964), Nigam (1972) and Nigam & Wilson (1980), noting that in such cases the finite size of atoms results in forbidden ranges of positional parameters, have shown that there is a diminution of the intensity of layers (rows) in the immediate neighbourhood of the enhanced zones (rows), just sufficient to compensate for the enhancement. In forming general averages, therefore, reflections from enhanced zones or rows should be included at their full intensity, not divided by the multiplier; the

matter is discussed in more detail by Wilson (1987a). It should be noted, however, that organic structures containing molecules related by rotation axes are rare, and such structures related by mirror planes are even rarer (Wilson, 1993).

### 2.1.3.3. *More than one symmetry element*

Further alterations of the intensities occur if two or more such symmetry elements are present in the space group. The effects were treated in detail by Rogers (1950), who used them to construct a table for the determination of space groups by supplementing the usual knowledge of Laue group with statistical information. Only two pairs of space groups, the orthorhombic *I222* and *I2<sub>1</sub>2<sub>1</sub>2<sub>1</sub>*, and their cubic supergroups *I23* and *I2<sub>1</sub>3<sub>1</sub>*, remained unresolved. Examination of this table shows that what statistical information does is to resolve the Laue group into point groups; the further resolution into space groups is equivalent to the use of Table 3.2 in *IT A* (1983). The statistical consequences of each point group, as given by Rogers, are reproduced in Table 2.1.3.3.

### 2.1.4. Probability density distributions – mathematical preliminaries

For the purpose of this chapter, ‘ideal’ probability distributions or probability density functions are the asymptotic forms obtained by the use of the central-limit theorem when the number of atoms in the unit cell, *N*, is sufficiently large. In order to derive them it is necessary to outline the properties of characteristic functions and to state alternative conditions for the validity of the central-limit theorem; the distributions themselves are derived in Section 2.1.5.

#### 2.1.4.1. *Characteristic functions*

The average value of  $\exp(itx)$  is very important in probability theory; it is called the characteristic function of the distribution  $f(x)$  and is denoted by  $C_x(t)$  or, when no confusion can arise, by  $C(t)$ . It exists for all legitimate distributions, whether discrete or continuous. In the continuous case it is given by

$$C(t) = \int_{-\infty}^{\infty} \exp(itx)f(x) dx, \quad (2.1.4.1)$$

and is thus the Fourier transform of  $f(x)$ . In many cases it can be obtained from known integrals. For example, for the Cauchy distribution,

$$C(t) = \frac{a}{\pi} \int_{-\infty}^{\infty} \frac{\exp(itx)}{a^2 + x^2} dx \quad (2.1.4.2)$$

$$= \exp(-a|t|), \quad (2.1.4.3)$$

and for the normal distribution,

$$C(t) = (2\pi\sigma^2)^{-1/2} \int_{-\infty}^{\infty} \exp\left(\frac{-(x-m)^2}{2\sigma^2}\right) \exp(itx) dx \quad (2.1.4.4)$$

$$= \exp\left(imt - \frac{\sigma^2 t^2}{2}\right). \quad (2.1.4.5)$$

Since the characteristic function is the Fourier transform of the distribution function, the converse is true, and if the characteristic function is known the probability distribution function can be obtained by the use of Fourier inversion theorem,

$$f(x) = (1/2\pi) \int_{-\infty}^{\infty} \exp(-itx)C(t) dt. \quad (2.1.4.6)$$