

## 1.2. THE STRUCTURE FACTOR

Table 1.2.7.3 (cont.)

(d) Index rules for cubic symmetries (Kurki-Suonio, 1977; Kara &amp; Kurki-Suonio, 1981).

$l$	$j$	$23$ $T$	$m\bar{3}$ $T_h$	$432$ $O$	$\bar{4}3m$ $T_d$	$m\bar{3}m$ $O_h$
0	1	×	×	×	×	×
3	1	×			×	
4	1	×	×	×	×	×
6	1	×	×	×	×	×
6	2	×	×			
7	1	×			×	
8	1	×	×	×	×	×
9	1	×			×	
9	2	×		×		
10	1	×	×	×	×	×
10	2	×	×			

respectively, where it has been assumed that the radial function depends only on  $l$ .

Because the spherical harmonic functions form a complete set, their products can be expressed as a linear combination of spherical harmonics. The coefficients in this expansion are the *Clebsch–Gordan coefficients* (Condon & Shortley, 1957), defined by

$$Y_{lm}(\theta, \varphi)Y_{l'm'}(\theta, \varphi) = \sum_L \sum_M C_{LL'}^{Mmm'} Y_{LM}(\theta, \varphi) \quad (1.2.8.5a)$$

or the equivalent definition

$$C_{LL'}^{Mmm'} = \int_0^\pi \sin \theta \, d\theta \int_0^{2\pi} d\varphi Y_{LM}^*(\theta, \varphi) Y_{lm}(\theta, \varphi) Y_{l'm'}(\theta, \varphi). \quad (1.2.8.5b)$$

The  $C_{LL'}^{Mmm'}$  vanish, unless  $L + l + l'$  is even,  $|l - l'| < L < l + l'$  and  $M = m + m'$ .

The corresponding expression for  $y_{lmp}$  is

$$y_{lmp}(\theta, \varphi)y_{l'm'p'}(\theta, \varphi) = \sum_L \sum_M C'_{LL'}^{Mmm'} y_{LMP}(\theta, \varphi), \quad (1.2.8.5c)$$

with  $M = |m + m'|$  and  $|m - m'|$  for  $p = p'$ , and  $M = -|m + m'|$  and  $-|m - m'|$  for  $p = -p'$  and  $P = p \times p'$ .

Values of  $C$  and  $C'$  for  $l \leq 2$  are given in Tables 1.2.8.1 and 1.2.8.2. They are valid for the functions  $Y_{lm}$  and  $y_{lmp}$  with normalization  $\int |Y_{lm}|^2 d\Omega = 1$  and  $\int y_{lmp}^2 d\Omega = 1$ .

By using (1.2.8.5a) or (1.2.8.5c), the one-centre orbital products are expressed as a sum of spherical harmonic functions. It follows that the one-centre orbital product density basis set is formally equivalent to the multipole description, both in real and in reciprocal space. To obtain the relation between orbital products and the charge-density functions, the right-hand side of (1.2.8.5c) has to be multiplied by the ratio of the normalization constants, as the wavefunctions  $y_{lmp}$  and charge-density functions  $d_{lmp}$  are normalized in a different way as described by (1.2.7.3a) and (1.2.7.3b). Thus

$$y_{lmp}(\theta, \varphi)y_{l'm'p'}(\theta, \varphi) = \sum_L \sum_M R_{LMP} C'_{LL'}^{Mmm'} d_{LMP}(\theta, \varphi), \quad (1.2.8.6)$$

where  $R_{LMP} = M_{LMP}$  (wavefunction)/ $L_{LMP}$  (density function). The normalization constants  $M_{lmp}$  and  $L_{lmp}$  are given in Table 1.2.7.1, while the coefficients in the expressions (1.2.8.6) are listed in Table 1.2.8.3.

### 1.2.8.2. Two-centre orbital products

Fourier transform of the electron density as described by (1.2.8.1) requires explicit expressions for the two-centre orbital product scattering. Such expressions are described in the literature for both Gaussian (Stewart, 1969b) and Slater-type (Bentley

& Stewart, 1973; Avery & Ørmen, 1979) atomic orbitals. The expressions can also be used for Hartree–Fock atomic functions, as expansions in terms of Gaussian- (Stewart, 1969b, 1970; Stewart & Hehre, 1970; Hehre *et al.*, 1970) and Slater-type (Clementi & Roetti, 1974) functions are available for many atoms.

### 1.2.9. The atomic temperature factor

Since the crystal is subject to vibrational oscillations, the observed elastic scattering intensity is an average over all normal modes of the crystal. Within the Born–Oppenheimer approximation, the theoretical electron density should be calculated for each set of nuclear coordinates. An average can be obtained by taking into account the statistical weight of each nuclear configuration, which may be expressed by the probability distribution function  $P(\mathbf{u}_1, \dots, \mathbf{u}_N)$  for a set of displacement coordinates  $\mathbf{u}_1, \dots, \mathbf{u}_N$ .

In general, if  $\rho(\mathbf{r}, \mathbf{u}_1, \dots, \mathbf{u}_N)$  is the electron density corresponding to the geometry defined by  $\mathbf{u}_1, \dots, \mathbf{u}_N$ , the time-averaged electron density is given by

$$\langle \rho(\mathbf{r}) \rangle = \int \rho(\mathbf{r}, \mathbf{u}_1, \dots, \mathbf{u}_N) P(\mathbf{u}_1, \dots, \mathbf{u}_N) d\mathbf{u}_1 \dots d\mathbf{u}_N. \quad (1.2.9.1)$$

When the crystal can be considered as consisting of perfectly following rigid entities, which may be molecules or atoms, expression (1.2.9.1) simplifies:

$$\langle \rho_{\text{rigid group}}(\mathbf{r}) \rangle = \int \rho_{\text{r.g., static}}(\mathbf{r} - \mathbf{u}) P(\mathbf{u}) d\mathbf{u} = \rho_{\text{r.g., static}} * P(\mathbf{u}). \quad (1.2.9.2)$$

In the approximation that the atomic electrons perfectly follow the nuclear motion, one obtains

$$\langle \rho_{\text{atom}}(\mathbf{r}) \rangle = \rho_{\text{atom, static}}(\mathbf{r}) * P(\mathbf{u}). \quad (1.2.9.3)$$

The Fourier transform of this convolution is the product of the Fourier transforms of the individual functions:

$$\langle f(\mathbf{H}) \rangle = f(\mathbf{H})T(\mathbf{H}). \quad (1.2.9.4)$$

Thus  $T(\mathbf{H})$ , the atomic temperature factor, is the Fourier transform of the probability distribution  $P(\mathbf{u})$ .

### 1.2.10. The vibrational probability distribution and its Fourier transform in the harmonic approximation

For a harmonic oscillator, the probability distribution averaged over all populated energy levels is a Gaussian, centred at the equilibrium position. For the three-dimensional isotropic harmonic oscillator, the distribution is

# 1. GENERAL RELATIONSHIPS AND TECHNIQUES

Table 1.2.7.4. Closed-form expressions for Fourier transform of Slater-type functions (Avery & Watson, 1977; Su & Coppens, 1990)

$$\langle j_k \rangle \equiv \int_0^\infty r^N \exp(-Zr) j_k(Kr) dr, K = 4\pi \sin \theta / \lambda.$$

k	N							
	1	2	3	4	5	6	7	8
0	$\frac{1}{K^2 + Z^2}$	$\frac{2Z}{(K^2 + Z^2)^2}$	$\frac{2(3Z^2 - K^2)}{(K^2 + Z^2)^3}$	$\frac{24Z(Z^2 - K^2)}{(K^2 + Z^2)^4}$	$\frac{24(5Z^2 - 10K^2Z^2 + K^4)}{(K^2 + Z^2)^5}$	$\frac{240Z(K^2 - 3Z^2)(3K^2 - Z^2)}{(K^2 + Z^2)^6}$	$\frac{720(7Z^6 - 35K^2Z^4 + 21K^4Z^2 - K^6)}{(K^2 + Z^2)^7}$	$\frac{40320(Z^7 - 7K^2Z^5 + 7K^4Z^3 - K^6Z)}{(K^2 + Z^2)^8}$
1		$\frac{2K}{(K^2 + Z^2)^2}$	$\frac{8KZ}{(K^2 + Z^2)^3}$	$\frac{8K(5Z^2 - K^2)}{(K^2 + Z^2)^4}$	$\frac{48KZ(5Z^2 - 3K^2)}{(K^2 + Z^2)^5}$	$\frac{48K(35Z^4 - 42K^2Z^2 + 3K^4)}{(K^2 + Z^2)^6}$	$\frac{1920KZ(7Z^4 - 14K^2Z^2 + 3K^4)}{(K^2 + Z^2)^7}$	$\frac{5760K(21Z^6 - 63K^2Z^4 + 27K^4Z^2 - K^6)}{(K^2 + Z^2)^8}$
2			$\frac{8K^2}{(K^2 + Z^2)^3}$	$\frac{48K^2Z}{(K^2 + Z^2)^4}$	$\frac{48K^2(7Z^2 - K^2)}{(K^2 + Z^2)^5}$	$\frac{384K^2Z(7Z^2 - 3K^2)}{(K^2 + Z^2)^6}$	$\frac{1152K^2(21Z^4 - 18K^2Z^2 + K^4)}{(K^2 + Z^2)^7}$	$\frac{11520K^2Z(21Z^4 - 30K^2Z^2 + 5K^4)}{(K^2 + Z^2)^8}$
3				$\frac{48K^3}{(K^2 + Z^2)^4}$	$\frac{384K^3Z}{(K^2 + Z^2)^5}$	$\frac{384K^3(9Z^2 - K^2)}{(K^2 + Z^2)^6}$	$\frac{11520K^3Z(3Z^2 - K^2)}{(K^2 + Z^2)^7}$	$\frac{11520K^3(33Z^4 - 22K^2Z^2 + K^4)}{(K^2 + Z^2)^8}$
4					$\frac{384K^4}{(K^2 + Z^2)^5}$	$\frac{3840K^4Z}{(K^2 + Z^2)^6}$	$\frac{3840K^4(11Z^2 - K^2)}{(K^2 + Z^2)^7}$	$\frac{46080K^4Z(11Z^2 - 3K^2)}{(K^2 + Z^2)^8}$
5						$\frac{3840K^5}{(K^2 + Z^2)^6}$	$\frac{46080K^5Z}{(K^2 + Z^2)^7}$	$\frac{40680K^5(13Z^2 - K^2)}{(K^2 + Z^2)^8}$
6							$\frac{46080K^6}{(K^2 + Z^2)^7}$	$\frac{645120K^6Z}{(K^2 + Z^2)^8}$
7								$\frac{645120K^7}{(K^2 + Z^2)^8}$

$$P(\mathbf{u}) = (2\pi \langle u^2 \rangle)^{-3/2} \exp\{-|\mathbf{u}|^2 / 2 \langle u^2 \rangle\}, \quad (1.2.10.1)$$

where  $\langle u^2 \rangle$  is the mean-square displacement in any direction.

The corresponding trivariate normal distribution to be used for anisotropic harmonic motion is, in tensor notation,

$$P(\mathbf{u}) = \frac{|\sigma^{-1}|^{1/2}}{(2\pi)^{3/2}} \exp\{-\frac{1}{2} \sigma_{jk}^{-1} (u^j u^k)\}. \quad (1.2.10.2a)$$

Here  $\sigma$  is the variance-covariance matrix, with covariant components, and  $|\sigma^{-1}|$  is the determinant of the inverse of  $\sigma$ . Summation over repeated indices has been assumed. The corresponding equation in matrix notation is

$$P(\mathbf{u}) = \frac{|\sigma^{-1}|^{1/2}}{(2\pi)^{3/2}} \exp\{-\frac{1}{2} (\mathbf{u})^T \sigma^{-1} (\mathbf{u})\}, \quad (1.2.10.2b)$$

where the superscript  $T$  indicates the transpose.

The characteristic function, or Fourier transform, of  $P(\mathbf{u})$  is

$$T(\mathbf{H}) = \exp\{-2\pi^2 \sigma^{jk} h_j h_k\} \quad (1.2.10.3a)$$

or

$$T(\mathbf{H}) = \exp\{-2\pi^2 \mathbf{H}^T \sigma \mathbf{H}\}. \quad (1.2.10.3b)$$

With the change of variable  $b^{jk} = 2\pi^2 \sigma^{jk}$ , (1.2.10.3a) becomes

$$T(\mathbf{H}) = \exp\{-b^{jk} h_j h_k\}.$$

## 1.2.11. Rigid-body analysis

The treatment of rigid-body motion of molecules or molecular fragments was developed by Cruickshank (1956) and expanded into a general theory by Schomaker & Trueblood (1968). The theory has been described by Johnson (1970b) and by Dunitz (1979). The latter reference forms the basis for the following treatment.

The most general motions of a rigid body consist of rotations about three axes, coupled with translations parallel to each of the axes. Such motions correspond to screw rotations. A libration

around a vector  $\lambda$  ( $\lambda_1, \lambda_2, \lambda_3$ ), with length corresponding to the magnitude of the rotation, results in a displacement  $\delta \mathbf{r}$ , such that

$$\delta \mathbf{r} = (\lambda \times \mathbf{r}) = \mathbf{D} \mathbf{r} \quad (1.2.11.1)$$

with

$$\mathbf{D} = \begin{bmatrix} 0 & -\lambda_3 & \lambda_2 \\ \lambda_3 & 0 & -\lambda_1 \\ -\lambda_2 & \lambda_1 & 0 \end{bmatrix}, \quad (1.2.11.2)$$

or in Cartesian tensor notation, assuming summation over repeated indices,

$$\delta r_i = D_{ij} r_j = -\varepsilon_{ijk} \lambda_k r_j \quad (1.2.11.3)$$

where the permutation operator  $\varepsilon_{ijk}$  equals +1 for  $i, j, k$  a cyclic permutation of the indices 1, 2, 3, or -1 for a noncyclic permutation, and zero if two or more indices are equal. For  $i = 1$ , for example, only the  $\varepsilon_{123}$  and  $\varepsilon_{132}$  terms occur. Addition of a translational displacement gives

$$\delta r_i = D_{ij} r_j + t_i. \quad (1.2.11.4)$$

When a rigid body undergoes vibrations the displacements vary with time, so suitable averages must be taken to derive the mean-square displacements. If the librational and translational motions are independent, the cross products between the two terms in (1.2.11.4) average to zero and the elements of the mean-square displacement tensor of atom  $n$ ,  $U_{ij}^n$ , are given by

$$\begin{aligned} U_{11}^n &= +L_{22} r_3^2 + L_{33} r_2^2 - 2L_{23} r_2 r_3 + T_{11} \\ U_{22}^n &= +L_{33} r_1^2 + L_{11} r_3^2 - 2L_{13} r_1 r_3 + T_{22} \\ U_{33}^n &= +L_{11} r_2^2 + L_{22} r_1^2 - 2L_{12} r_1 r_2 + T_{33} \\ U_{12}^n &= -L_{33} r_1 r_2 - L_{12} r_3^2 + L_{13} r_2 r_3 + L_{23} r_1 r_3 + T_{12} \\ U_{13}^n &= -L_{22} r_1 r_3 + L_{12} r_2 r_3 - L_{13} r_2^2 + L_{23} r_1 r_2 + T_{13} \\ U_{23}^n &= -L_{11} r_2 r_3 + L_{12} r_1 r_3 - L_{13} r_1 r_2 - L_{23} r_1^2 + T_{23}, \end{aligned} \quad (1.2.11.5)$$

where the coefficients  $L_{ij} = \langle \lambda_i \lambda_j \rangle$  and  $T_{ij} = \langle t_i t_j \rangle$  are the elements of the  $3 \times 3$  libration tensor  $\mathbf{L}$  and the  $3 \times 3$  translation tensor  $\mathbf{T}$ , respectively. Since pairs of terms such as  $\langle t_i t_j \rangle$  and  $\langle t_j t_i \rangle$