International Tables for Crystallography (2010). Vol. B, Section 1.3.2.2.5, p. 27.

$$\|\mathbf{f}\|_p = \left(\int_{\mathbb{R}^n} |f(\mathbf{x})|^p \,\mathrm{d}^n\mathbf{x}\right)^{1/p} < \infty$$

are then complete for the topology induced by the norm $\|.\|_p$: the limit of every Cauchy sequence of functions in L^p is itself a function in L^p (Riesz–Fischer theorem).

The space $L^1(\mathbb{R}^n)$ consists of those function classes f such that

$$\|f\|_1 = \int_{\mathbb{R}^n} |f(\mathbf{x})| \,\mathrm{d}^n\mathbf{x} < \infty$$

which are called *summable* or *absolutely integrable*. The convolution product:

$$\begin{aligned}(f * g)(\mathbf{x}) &= \int_{\mathbb{R}^n} f(\mathbf{y}) g(\mathbf{x} - \mathbf{y}) \,\mathrm{d}^n\mathbf{y} \\ &= \int_{\mathbb{R}^n} f(\mathbf{x} - \mathbf{y}) g(\mathbf{y}) \,\mathrm{d}^n\mathbf{y} = (g * f)(\mathbf{x})\end{aligned}$$

is well defined; combined with the vector space structure of L^1, it makes L^1 into a (commutative) *convolution algebra*. However, this algebra has no unit element: there is no $f \in L^1$ such that $f * g = g$ for all $g \in L^1$; it has only approximate units, *i.e.* sequences (f_ν) such that $f_\nu * g$ tends to g in the L^1 topology as $\nu \to \infty$. This is one of the starting points of distribution theory.

The space $L^2(\mathbb{R}^n)$ of *square-integrable* functions can be endowed with a scalar product

$$(f, g) = \int_{\mathbb{R}^n} \overline{f(\mathbf{x})} g(\mathbf{x}) \,\mathrm{d}^n\mathbf{x}$$

which makes it into a *Hilbert space*. The Cauchy–Schwarz inequality

$$|(f, g)| \le [(f, f)(g, g)]^{1/2}$$

generalizes the fact that the absolute value of the cosine of an angle is less than or equal to 1.

The space $L^\infty(\mathbb{R}^n)$ is defined as the space of functions f such that

$$\|f\|_\infty = \lim_{p\to\infty} \|f\|_p = \lim_{p\to\infty} \left(\int_{\mathbb{R}^n} |f(\mathbf{x})|^p \,\mathrm{d}^n\mathbf{x}\right)^{1/p} < \infty.$$

The quantity $\|f\|_\infty$ is called the 'essential sup norm' of f, as it is the smallest positive number which $|f(\mathbf{x})|$ exceeds only on a subset of zero measure in $\mathbb{R}^n$. A function $f \in L^\infty$ is called *essentially bounded*.

1.3.2.2.5. *Tensor products. Fubini's theorem*

Let $f \in L^1(\mathbb{R}^m)$, $g \in L^1(\mathbb{R}^n)$. Then the function

$$f \otimes g : (\mathbf{x}, \mathbf{y}) \longmapsto f(\mathbf{x}) g(\mathbf{y})$$

is called the *tensor product* of f and g, and belongs to $L^1(\mathbb{R}^m \times \mathbb{R}^n)$. The finite linear combinations of functions of the form $f \otimes g$ span a subspace of $L^1(\mathbb{R}^m \times \mathbb{R}^n)$ called the tensor product of $L^1(\mathbb{R}^m)$ and $L^1(\mathbb{R}^n)$ and denoted $L^1(\mathbb{R}^m) \otimes L^1(\mathbb{R}^n)$.

The integration of a general function over $\mathbb{R}^m \times \mathbb{R}^n$ may be accomplished in two steps according to *Fubini's theorem*. Given $F \in L^1(\mathbb{R}^m \times \mathbb{R}^n)$, the functions

$$\begin{aligned}F_1 &: \mathbf{x} \longmapsto \int_{\mathbb{R}^n} F(\mathbf{x}, \mathbf{y}) \,\mathrm{d}^n\mathbf{y} \\ F_2 &: \mathbf{y} \longmapsto \int_{\mathbb{R}^m} F(\mathbf{x}, \mathbf{y}) \,\mathrm{d}^m\mathbf{x}\end{aligned}$$

exist for almost all $\mathbf{x} \in \mathbb{R}^m$ and almost all $\mathbf{y} \in \mathbb{R}^n$, respectively, are integrable, and

$$\int_{\mathbb{R}^m \times \mathbb{R}^n} F(\mathbf{x}, \mathbf{y}) \,\mathrm{d}^m\mathbf{x} \,\mathrm{d}^n\mathbf{y} = \int_{\mathbb{R}^m} F_1(\mathbf{x}) \,\mathrm{d}^m\mathbf{x} = \int_{\mathbb{R}^n} F_2(\mathbf{y}) \,\mathrm{d}^n\mathbf{y}.$$

Conversely, if any one of the integrals

(i) $\displaystyle\int_{\mathbb{R}^m \times \mathbb{R}^n} |F(\mathbf{x}, \mathbf{y})| \,\mathrm{d}^m\mathbf{x} \,\mathrm{d}^n\mathbf{y}$

(ii) $\displaystyle\int_{\mathbb{R}^m} \left(\int_{\mathbb{R}^n} |F(\mathbf{x}, \mathbf{y})| \,\mathrm{d}^n\mathbf{y}\right) \mathrm{d}^m\mathbf{x}$

(iii) $\displaystyle\int_{\mathbb{R}^n} \left(\int_{\mathbb{R}^m} |F(\mathbf{x}, \mathbf{y})| \,\mathrm{d}^m\mathbf{x}\right) \mathrm{d}^n\mathbf{y}$

is finite, then so are the other two, and the identity above holds. It is then (and only then) permissible to change the order of integrations.

Fubini's theorem is of fundamental importance in the study of tensor products and convolutions of distributions.

1.3.2.2.6. *Topology in function spaces*

Geometric intuition, which often makes 'obvious' the topological properties of the real line and of ordinary space, cannot be relied upon in the study of function spaces: the latter are infinite-dimensional, and several inequivalent notions of convergence may exist. A careful analysis of topological concepts and of their interrelationship is thus a necessary prerequisite to the study of these spaces. The reader may consult Dieudonné (1969, 1970), Friedman (1970), Trèves (1967) and Yosida (1965) for detailed expositions.

1.3.2.2.6.1. *General topology*

Most topological notions are first encountered in the setting of *metric spaces*. A metric space E is a set equipped with a *distance function* d from $E \times E$ to the non-negative reals which satisfies:

(i)	$d(x, y) = d(y, x)$	$\forall x, y \in E$	(symmetry);
(ii)	$d(x, y) = 0$	iff $x = y$	(separation);
(iii)	$d(x, z) \le d(x, y) + d(y, z)$	$\forall x, y, z \in E$	(triangular inequality).

By means of d, the following notions can be defined: open balls, neighbourhoods; open and closed sets, interior and closure; convergence of sequences, continuity of mappings; Cauchy sequences and completeness; compactness; connectedness. They suffice for the investigation of a great number of questions in analysis and geometry (see *e.g.* Dieudonné, 1969).

Many of these notions turn out to depend only on the properties of the collection $\mathcal{O}(E)$ of open subsets of E: two distance functions leading to the same $\mathcal{O}(E)$ lead to identical topological properties. An axiomatic reformulation of topological notions is thus possible: a *topology* in E is a collection $\mathcal{O}(E)$ of subsets of E which satisfy suitable axioms and are deemed open irrespective

references