International Tables for Crystallography (2010). Vol. B, Section 1.3.2.3.3.2, p. 29.



$$\begin{aligned}\varphi(\mathbf{0}) &= \lim_{\nu\to\infty} \int_{\mathbb{R}^n} f_\nu(\mathbf{x})\varphi(\mathbf{x})\,\mathrm{d}^n\mathbf{x}\\ &\neq \int_{\mathbb{R}^n} \Big[\lim_{\nu\to\infty} f_\nu(\mathbf{x})\Big]\varphi(\mathbf{x})\,\mathrm{d}^n\mathbf{x} = 0\end{aligned}$$

because the sequence $(f_\nu)$ does not satisfy the hypotheses of Lebesgue's dominated convergence theorem.

Schwartz's solution to this problem is deceptively simple: the regular behaviour one is trying to capture is an attribute not of the sequence of functions $(f_\nu)$, but of the sequence of continuous linear functionals

$$T_\nu : \varphi \longmapsto \int_{\mathbb{R}^n} f_\nu(\mathbf{x})\varphi(\mathbf{x})\,\mathrm{d}^n\mathbf{x}$$

which has as a limit the continuous functional

$$T : \varphi \longmapsto \varphi(\mathbf{0}).$$

It is the latter functional which constitutes the proper definition of $\delta$. The previous paradoxes arose because one insisted on writing down the simple linear operation $T$ in terms of an integral.

The essence of Schwartz's theory of distributions is thus that, rather than try to define and handle 'generalized functions' *via* sequences such as $(f_\nu)$ [an approach adopted *e.g.* by Lighthill (1958) and Erdélyi (1962)], one should instead look at them as continuous linear functionals over spaces of well behaved functions.

There are many books on distribution theory and its applications. The reader may consult in particular Schwartz (1965, 1966), Gel'fand & Shilov (1964), Bremermann (1965), Trèves (1967), Challifour (1972), Friedlander (1982), and the relevant chapters of Hörmander (1963) and Yosida (1965). Schwartz (1965) is especially recommended as an introduction.

#### 1.3.2.3.2. *Rationale*

The guiding principle which leads to requiring that the functions $\varphi$ above (traditionally called 'test functions') should be well behaved is that correspondingly 'wilder' behaviour can then be accommodated in the limiting behaviour of the $f_\nu$ while still keeping the integrals $\int_{\mathbb{R}^n} f_\nu\varphi\,\mathrm{d}^n\mathbf{x}$ under control. Thus

(i) to minimize restrictions on the limiting behaviour of the $f_\nu$ *at infinity*, the $\varphi$'s will be chosen to have *compact support*;

(ii) to minimize restrictions on the *local* behaviour of the $f_\nu$, the $\varphi$'s will be chosen *infinitely differentiable*.

To ensure further the *continuity* of functionals such as $T_\nu$ with respect to the test function $\varphi$ as the $f_\nu$ go increasingly wild, very strong control will have to be exercised in the way in which a sequence $(\varphi_j)$ of test functions will be said to converge towards a limiting $\varphi$: conditions will have to be imposed not only on the values of the *functions* $\varphi_j$, but also on those of all their *derivatives*. Hence, defining a strong enough topology on the space of test functions $\varphi$ is an *essential* prerequisite to the development of a satisfactory theory of distributions.

#### 1.3.2.3.3. *Test-function spaces*

With this rationale in mind, the following function spaces will be defined for any open subset $\Omega$ of $\mathbb{R}^n$ (which may be the whole of $\mathbb{R}^n$):

(*a*) $\mathscr{E}(\Omega)$ is the space of complex-valued functions over $\Omega$ which are indefinitely differentiable;

(*b*) $\mathscr{D}(\Omega)$ is the subspace of $\mathscr{E}(\Omega)$ consisting of functions with (unspecified) compact support contained in $\mathbb{R}^n$;

(*c*) $\mathscr{D}_K(\Omega)$ is the subspace of $\mathscr{D}(\Omega)$ consisting of functions whose (compact) support is contained within a fixed compact subset $K$ of $\Omega$.

When $\Omega$ is unambiguously defined by the context, we will simply write $\mathscr{E}$, $\mathscr{D}$, $\mathscr{D}_K$.

It sometimes suffices to require the existence of continuous derivatives only up to finite order $m$ inclusive. The corresponding spaces are then denoted $\mathscr{E}^{(m)}$, $\mathscr{D}^{(m)}$, $\mathscr{D}_K^{(m)}$ with the convention that if $m = 0$, only continuity is required.

The *topologies* on these spaces constitute the most important ingredients of distribution theory, and will be outlined in some detail.

##### 1.3.2.3.3.1. *Topology on* $\mathscr{E}(\Omega)$

It is defined by the family of semi-norms

$$\varphi \in \mathscr{E}(\Omega) \longmapsto \sigma_{\mathbf{p},K}(\varphi) = \sup_{\mathbf{x}\in K} |D^{\mathbf{p}}\varphi(\mathbf{x})|,$$

where $\mathbf{p}$ is a multi-index and $K$ a compact subset of $\Omega$. A fundamental system $S$ of neighbourhoods of the origin in $\mathscr{E}(\Omega)$ is given by subsets of $\mathscr{E}(\Omega)$ of the form

$$V(m,\varepsilon,K) = \{\varphi \in \mathscr{E}(\Omega) \,|\, |\mathbf{p}| \le m \Rightarrow \sigma_{\mathbf{p},K}(\varphi) < \varepsilon\}$$

for all natural integers $m$, positive real $\varepsilon$, and compact subset $K$ of $\Omega$. Since a *countable* family of compact subsets $K$ suffices to cover $\Omega$, and since restricted values of $\varepsilon$ of the form $\varepsilon = 1/N$ lead to the same topology, $S$ is equivalent to a *countable* system of neighbourhoods and hence $\mathscr{E}(\Omega)$ *is metrizable.*

Convergence in $\mathscr{E}$ may thus be defined by means of sequences. A sequence $(\varphi_\nu)$ in $\mathscr{E}$ will be said to converge to 0 if for any given $V(m,\varepsilon,K)$ there exists $\nu_0$ such that $\varphi_\nu \in V(m,\varepsilon,K)$ whenever $\nu > \nu_0$; in other words, if the $\varphi_\nu$ and all their derivatives $D^{\mathbf{p}}\varphi_\nu$ converge to 0 uniformly on any given compact $K$ in $\Omega$.

##### 1.3.2.3.3.2. *Topology on* $\mathscr{D}_k(\Omega)$

It is defined by the family of semi-norms

$$\varphi \in \mathscr{D}_K(\Omega) \longmapsto \sigma_{\mathbf{p}}(\varphi) = \sup_{\mathbf{x}\in K} |D^{\mathbf{p}}\varphi(\mathbf{x})|,$$

where $K$ is now fixed. The fundamental system $S$ of neighbourhoods of the origin in $\mathscr{D}_K$ is given by sets of the form

$$V(m,\varepsilon) = \{\varphi \in \mathscr{D}_K(\Omega) \,|\, |\mathbf{p}| \le m \Rightarrow \sigma_{\mathbf{p}}(\varphi) < \varepsilon\}.$$

It is equivalent to the countable subsystem of the $V(m, 1/N)$, hence $\mathscr{D}_K(\Omega)$ *is metrizable.*

Convergence in $\mathscr{D}_K$ may thus be defined by means of sequences. A sequence $(\varphi_\nu)$ in $\mathscr{D}_K$ will be said to converge to 0 if for any given $V(m,\varepsilon)$ there exists $\nu_0$ such that $\varphi_\nu \in V(m,\varepsilon)$ whenever $\nu > \nu_0$; in other words, if the $\varphi_\nu$ and all their derivatives $D^{\mathbf{p}}\varphi_\nu$ converge to 0 uniformly in $K$.

##### 1.3.2.3.3.3. *Topology on* $\mathscr{D}(\Omega)$

It is defined by the fundamental system of neighbourhoods of the origin consisting of sets of the form

$$\begin{aligned}&V((m),(\varepsilon))\\ &\quad= \left\{\varphi \in \mathscr{D}(\Omega) \,|\, |\mathbf{p}| \le m_\nu \Rightarrow \sup_{\|\mathbf{x}\|\le\nu} |D^{\mathbf{p}}\varphi(\mathbf{x})| < \varepsilon_\nu \text{ for all } \nu\right\},\end{aligned}$$



references