where (m) is an increasing sequence (m_ν) of integers tending to $+\infty$ and (ε) is a decreasing sequence (ε_ν) of positive reals tending to 0, as $\nu \to \infty$.

This topology is *not metrizable*, because the sets of sequences (m) and (ε) are essentially uncountable. It can, however, be shown to be the *inductive limit* of the topology of the subspaces $\mathcal{D}_K$, in the following sense: V is a neighbourhood of the origin in $\mathcal{D}$ if and only if its intersection with $\mathcal{D}_K$ is a neighbourhood of the origin in $\mathcal{D}_K$ for any given compact K in Ω.

A sequence (φ_ν) in $\mathcal{D}$ will thus be said to converge to 0 in $\mathcal{D}$ if all the φ_ν belong to some $\mathcal{D}_K$ (with K a compact subset of Ω independent of ν) and if (φ_ν) converges to 0 in $\mathcal{D}_K$.

As a result, a complex-valued functional T on $\mathcal{D}$ will be said to be continuous for the topology of $\mathcal{D}$ if and only if, for any given compact K in Ω, its restriction to $\mathcal{D}_K$ is continuous for the topology of $\mathcal{D}_K$, *i.e.* maps convergent sequences in $\mathcal{D}_K$ to convergent sequences in $\mathbb{C}$.

This property of $\mathcal{D}$, *i.e.* having a nonmetrizable topology which is the inductive limit of metrizable topologies in its subspaces $\mathcal{D}_K$, conditions the whole structure of distribution theory and dictates that of many of its proofs.

1.3.2.3.3.4. Topologies on $\mathcal{E}^{(m)}$, $\mathcal{D}_k^{(m)}$, $\mathcal{D}^{(m)}$

These are defined similarly, but only involve conditions on derivatives up to order m.

1.3.2.3.4. Definition of distributions

A *distribution* T on Ω is a *linear form* over $\mathcal{D}(\Omega)$, *i.e.* a map

$$T : \varphi \longmapsto \langle T, \varphi \rangle$$

which associates linearly a complex number $\langle T, \varphi \rangle$ to any $\varphi \in \mathcal{D}(\Omega)$, and which is *continuous* for the topology of that space. In the terminology of Section 1.3.2.2.6.2, T is an element of $\mathcal{D}'(\Omega)$, the *topological dual* of $\mathcal{D}(\Omega)$.

Continuity over $\mathcal{D}$ is equivalent to continuity over $\mathcal{D}_K$ for all compact K contained in Ω, and hence to the condition that for any sequence (φ_ν) in $\mathcal{D}$ such that

(i) Supp φ_ν is contained in some compact K independent of ν,

(ii) the sequences $(|D^{\mathbf{p}}\varphi_\nu|)$ converge uniformly to 0 on K for all multi-indices $\mathbf{p}$;

then the sequence of complex numbers $\langle T, \varphi_\nu \rangle$ converges to 0 in $\mathbb{C}$.

If the continuity of a distribution T requires (ii) for $|\mathbf{p}| \leq m$ only, T may be defined over $\mathcal{D}^{(m)}$ and thus $T \in \mathcal{D}'^{(m)}$; T is said to be a *distribution of finite order m*. In particular, for $m = 0$, $\mathcal{D}^{(0)}$ is the space of continuous functions with compact support, and a distribution $T \in \mathcal{D}'^{(0)}$ is a (Radon) *measure* as used in the theory of integration. Thus measures are particular cases of distributions.

Generally speaking, the *larger* a space of test functions, the *smaller* its topological dual:

$$m < n \Rightarrow \mathcal{D}^{(m)} \supset \mathcal{D}^{(n)} \Rightarrow \mathcal{D}'^{(n)} \supset \mathcal{D}'^{(m)}.$$

This clearly results from the observation that if the φ's are allowed to be less regular, then less wildness can be accommodated in T if the continuity of the map $\varphi \longmapsto \langle T, \varphi \rangle$ with respect to φ is to be preserved.

1.3.2.3.5. First examples of distributions

(i) The linear map $\varphi \longmapsto \langle \delta, \varphi \rangle = \varphi(\mathbf{0})$ is a measure (*i.e.* a zeroth-order distribution) called Dirac's measure or (improperly) Dirac's 'δ-function'.

(ii) The linear map $\varphi \longmapsto \langle \delta_{(\mathbf{a})}, \varphi \rangle = \varphi(\mathbf{a})$ is called Dirac's measure at point $\mathbf{a} \in \mathbb{R}^n$.

(iii) The linear map $\varphi \longmapsto (-1)^{\mathbf{p}} D^{\mathbf{p}}\varphi(\mathbf{a})$ is a distribution of order $m = |\mathbf{p}| > 0$, and hence is not a measure.

(iv) The linear map $\varphi \longmapsto \sum_{\nu>0} \varphi^{(\nu)}(\nu)$ is a distribution of infinite order on $\mathbb{R}$: the order of differentiation is bounded for each φ (because φ has compact support) but is not as φ varies.

(v) If $(\mathbf{p}_\nu)$ is a sequence of multi-indices $\mathbf{p}_\nu = (p_{1\nu}, \ldots, p_{n\nu})$ such that $|\mathbf{p}_\nu| \to \infty$ as $\nu \to \infty$, then the linear map $\varphi \longmapsto \sum_{\nu>0} (D^{\mathbf{p}_\nu}\varphi)(\mathbf{p}_\nu)$ is a distribution of infinite order on $\mathbb{R}^n$.

1.3.2.3.6. Distributions associated to locally integrable functions

Let f be a complex-valued function over Ω such that $\int_K |f(\mathbf{x})| \, \mathrm{d}^n\mathbf{x}$ exists for any given compact K in Ω; f is then called *locally integrable*.

The linear mapping from $\mathcal{D}(\Omega)$ to $\mathbb{C}$ defined by

$$\varphi \longmapsto \int_\Omega f(\mathbf{x})\varphi(\mathbf{x}) \, \mathrm{d}^n\mathbf{x}$$

may then be shown to be continuous over $\mathcal{D}(\Omega)$. It thus defines a *distribution* $T_f \in \mathcal{D}'(\Omega)$:

$$\langle T_f, \varphi \rangle = \int_\Omega f(\mathbf{x})\varphi(\mathbf{x}) \, \mathrm{d}^n\mathbf{x}.$$

As the continuity of T_f only requires that $\varphi \in \mathcal{D}^{(0)}(\Omega)$, T_f is actually a Radon measure.

It can be shown that two locally integrable functions f and g define the same distribution, *i.e.*

$$\langle T_f, \varphi \rangle = \langle T_K, \varphi \rangle \quad \text{for all } \varphi \in \mathcal{D},$$

if and only if they are equal almost everywhere. The classes of locally integrable functions modulo this equivalence form a vector space denoted $L^1_{\text{loc}}(\Omega)$; each element of $L^1_{\text{loc}}(\Omega)$ may therefore be identified with the distribution T_f defined by any one of its representatives f.

1.3.2.3.7. Support of a distribution

A distribution $T \in \mathcal{D}'(\Omega)$ is said to *vanish* on an open subset ω of Ω if it vanishes on all functions in $\mathcal{D}(\omega)$, *i.e.* if $\langle T, \varphi \rangle = 0$ whenever $\varphi \in \mathcal{D}(\omega)$.

The *support* of a distribution T, denoted Supp T, is then defined as the complement of the set-theoretic union of those open subsets ω on which T vanishes; or equivalently as the smallest closed subset of Ω outside which T vanishes.

When $T = T_f$ for $f \in L^1_{\text{loc}}(\Omega)$, then Supp T = Supp f, so that the two notions coincide. Clearly, if Supp T and Supp φ are disjoint subsets of Ω, then $\langle T, \varphi \rangle = 0$.

It can be shown that any distribution $T \in \mathcal{D}'$ with compact support may be extended from $\mathcal{D}$ to $\mathcal{E}$ while remaining continuous, so that $T \in \mathcal{E}'$; and that conversely, if $S \in \mathcal{E}'$, then its restriction T to $\mathcal{D}$ is a distribution with compact support. Thus, *the topological dual $\mathcal{E}'$ of $\mathcal{E}$ consists of those distributions in $\mathcal{D}'$ which have compact support*. This is intuitively clear since, if the condition of having compact support is fulfilled by T, it needs no longer be required of φ, which may then roam through $\mathcal{E}$ rather than $\mathcal{D}$.

1.3.2.3.8. Convergence of distributions

A *sequence* (T_j) of distributions will be said to converge in $\mathcal{D}'$ to a distribution T as $j \to \infty$ if, for any given $\varphi \in \mathcal{D}$, the sequence of complex numbers $(\langle T_j, \varphi \rangle)$ converges in $\mathbb{C}$ to the complex number $\langle T, \varphi \rangle$.

A *series* $\sum_{j=0}^{\infty} T_j$ of distributions will be said to converge in $\mathcal{D}'$ and to have distribution S as its sum if the sequence of partial sums $S_k = \sum_{j=0}^{k}$ converges to S.