$$\mathcal{F}[f * g](\xi) = \int_{\mathbb{R}^n} \left[\int_{\mathbb{R}^n} f(\mathbf{y}) g(\mathbf{x} - \mathbf{y}) \, \mathrm{d}^n\mathbf{y} \right] \exp(-2\pi i \xi \cdot \mathbf{x}) \, \mathrm{d}^n\mathbf{x}.$$

With $\mathbf{x} = \mathbf{y} + \mathbf{z}$, so that

$$\exp(-2\pi i \xi \cdot \mathbf{x}) = \exp(-2\pi i \xi \cdot \mathbf{y}) \exp(-2\pi i \xi \cdot \mathbf{z}),$$

and with Fubini's theorem, rearrangement of the double integral gives:

$$\mathcal{F}[f * g] = \mathcal{F}[f] \times \mathcal{F}[g]$$

and similarly

$$\bar{\mathcal{F}}[f * g] = \bar{\mathcal{F}}[f] \times \bar{\mathcal{F}}[g].$$

Thus *the Fourier transform and cotransform turn convolution into multiplication*.

1.3.2.4.2.6. Reciprocity property

In general, $\mathcal{F}[f]$ and $\bar{\mathcal{F}}[f]$ are not summable, and hence cannot be further transformed; however, as they are essentially bounded, their products with the Gaussians $G_t(\xi) = \exp(-2\pi^2 \|\xi\|^2 t)$ are summable for all $t > 0$, and it can be shown that

$$f = \lim_{t \to 0} \bar{\mathcal{F}}[G_t \mathcal{F}[f]] = \lim_{t \to 0} \mathcal{F}[G_t \bar{\mathcal{F}}[f]],$$

where the limit is taken in the topology of the L^1 norm $\|.\|_1$. Thus $\mathcal{F}$ and $\bar{\mathcal{F}}$ are (in a sense) mutually inverse, which justifies the common practice of calling $\bar{\mathcal{F}}$ the 'inverse Fourier transformation'.

1.3.2.4.2.7. Riemann–Lebesgue lemma

If $f \in L^1(\mathbb{R}^n)$, *i.e.* is summable, then $\mathcal{F}[f]$ and $\bar{\mathcal{F}}[f]$ exist and are continuous and essentially bounded:

$$\|\mathcal{F}[f]\|_\infty = \|\bar{\mathcal{F}}[f]\|_\infty \leq \|f\|_1.$$

In fact one has the much stronger property, whose statement constitutes the *Riemann–Lebesgue lemma*, that $\mathcal{F}[f](\xi)$ and $\bar{\mathcal{F}}[f](\xi)$ both tend to zero as $\|\xi\| \to \infty$.

1.3.2.4.2.8. Differentiation

Let us now suppose that $n = 1$ and that $f \in L^1(\mathbb{R})$ is differentiable with $f' \in L^1(\mathbb{R})$. Integration by parts yields

$$\begin{aligned}\mathcal{F}[f'](\xi) &= \int_{-\infty}^{+\infty} f'(x) \exp(-2\pi i \xi \cdot x) \, \mathrm{d}x \\ &= [f(x) \exp(-2\pi i \xi \cdot x)]_{-\infty}^{+\infty} \\ &\quad + 2\pi i \xi \int_{-\infty}^{+\infty} f(x) \exp(-2\pi i \xi \cdot x) \, \mathrm{d}x.\end{aligned}$$

Since f' is summable, f has a limit when $x \to \pm\infty$, and this limit must be 0 since f is summable. Therefore

$$\mathcal{F}[f'](\xi) = (2\pi i \xi) \mathcal{F}[f](\xi)$$

with the bound

$$\|2\pi\xi \mathcal{F}[f]\|_\infty \leq \|f'\|_1$$

so that $|\mathcal{F}[f](\xi)|$ decreases faster than $1/|\xi| \to \infty$.

This result can be easily extended to several dimensions and to any multi-index $\mathbf{m}$: if f is summable and has continuous summable partial derivatives up to order $|\mathbf{m}|$, then

$$\mathcal{F}[D^{\mathbf{m}} f](\xi) = (2\pi i \xi)^{\mathbf{m}} \mathcal{F}[f](\xi)$$

and

$$\|(2\pi\xi)^{\mathbf{m}} \mathcal{F}[f]\|_\infty \leq \|D^{\mathbf{m}} f\|_1.$$

Similar results hold for $\bar{\mathcal{F}}$, with $2\pi i \xi$ replaced by $-2\pi i \xi$. Thus, the more differentiable f is, with summable derivatives, the faster $\mathcal{F}[f]$ and $\bar{\mathcal{F}}[f]$ decrease at infinity.

The property of turning differentiation into multiplication by a monomial has many important applications in crystallography, for instance differential syntheses (Sections 1.3.4.2.1.9, 1.3.4.4.7.2, 1.3.4.4.7.5) and moment-generating functions [Section 1.3.4.5.2.1(*c*)].

1.3.2.4.2.9. Decrease at infinity

Conversely, assume that f is summable on $\mathbb{R}^n$ and that f decreases fast enough at infinity for $\mathbf{x}^{\mathbf{m}} f$ also to be summable, for some multi-index $\mathbf{m}$. Then the integral defining $\mathcal{F}[f]$ may be subjected to the differential operator $D^{\mathbf{m}}$, still yielding a convergent integral: therefore $D^{\mathbf{m}} \mathcal{F}[f]$ exists, and

$$D^{\mathbf{m}}(\mathcal{F}[f])(\xi) = \mathcal{F}[(-2\pi i \mathbf{x})^{\mathbf{m}} f](\xi)$$

with the bound

$$\|D^{\mathbf{m}}(\mathcal{F}[f])\|_\infty = \|(2\pi\mathbf{x})^{\mathbf{m}} f\|_1.$$

Similar results hold for $\bar{\mathcal{F}}$, with $-2\pi i \mathbf{x}$ replaced by $2\pi i \mathbf{x}$. Thus, the faster f decreases at infinity, the more $\mathcal{F}[f]$ and $\bar{\mathcal{F}}[f]$ are differentiable, with bounded derivatives. This property is the converse of that described in Section 1.3.2.4.2.8, and their combination is fundamental in the definition of the function space $\mathcal{S}$ in Section 1.3.2.4.4.1, of tempered distributions in Section 1.3.2.5, and in the extension of the Fourier transformation to them.

1.3.2.4.2.10. The Paley–Wiener theorem

An extreme case of the last instance occurs when f has *compact support*: then $\mathcal{F}[f]$ and $\bar{\mathcal{F}}[f]$ are so regular that they may be analytically continued from $\mathbb{R}^n$ to $\mathbb{C}^n$ where they are *entire* functions, *i.e.* have no singularities at finite distance (Paley & Wiener, 1934). This is easily seen for $\mathcal{F}[f]$: giving vector $\boldsymbol{\xi} \in \mathbb{R}^n$ a vector $\boldsymbol{\eta} \in \mathbb{R}^n$ of imaginary parts leads to

$$\begin{aligned}\mathcal{F}[f](\boldsymbol{\xi} + i\boldsymbol{\eta}) &= \int_{\mathbb{R}^n} f(\mathbf{x}) \exp[-2\pi i(\boldsymbol{\xi} + i\boldsymbol{\eta}) \cdot \mathbf{x}] \, \mathrm{d}^n\mathbf{x} \\ &= \mathcal{F}[\exp(2\pi\boldsymbol{\eta} \cdot \mathbf{x}) f](\boldsymbol{\xi}),\end{aligned}$$

where the latter transform always exists since $\exp(2\pi\boldsymbol{\eta} \cdot \mathbf{x}) f$ is summable with respect to $\mathbf{x}$ for all values of $\boldsymbol{\eta}$. This analytic continuation forms the basis of the saddlepoint method in probability theory [Section 1.3.4.5.2.1(*f*)] and leads to the use of maximum-entropy distributions in the statistical theory of direct phase determination [Section 1.3.4.5.2.2(*e*)].