International Tables for Crystallography (2010). Vol. B, Section 1.3.2.5.4, p. 40.

1.3.2.5.2. $\mathscr{S}$ as a test-function space

A notion of convergence has to be introduced in $\mathscr{S}(\mathbb{R}^n)$ in order to be able to define and test the continuity of linear functionals on it.

A sequence (φ_j) of functions in $\mathscr{S}$ will be said to converge to 0 if, for any given multi-indices $\mathbf{k}$ and $\mathbf{p}$, the sequence $(\mathbf{x}^{\mathbf{k}} D^{\mathbf{p}} \varphi_j)$ tends to 0 uniformly on $\mathbb{R}^n$.

It can be shown that $\mathscr{D}(\mathbb{R}^n)$ is dense in $\mathscr{S}(\mathbb{R}^n)$. Translation is continuous for this topology. For any linear differential operator $P(D) = \sum_{\mathbf{p}} a_{\mathbf{p}} D^{\mathbf{p}}$ and any polynomial $Q(\mathbf{x})$ over $\mathbb{R}^n$, $(\varphi_j) \rightarrow 0$ implies $[Q(\mathbf{x}) \times P(D)\varphi_j] \rightarrow 0$ in the topology of $\mathscr{S}$. Therefore, differentiation and multiplication by polynomials are continuous for the topology on $\mathscr{S}$.

The Fourier transformations $\mathscr{F}$ and $\bar{\mathscr{F}}$ are also continuous for the topology of $\mathscr{S}$. Indeed, let (φ_j) converge to 0 for the topology on $\mathscr{S}$. Then, by Section 1.3.2.4.2,

$$\|(2\pi\boldsymbol{\xi})^{\mathbf{m}} D^{\mathbf{p}}(\mathscr{F}[\varphi_j])\|_\infty \leq \|D^{\mathbf{m}}[(2\pi\mathbf{x})^{\mathbf{p}}\varphi_j]\|_1.$$

The right-hand side tends to 0 as $j \rightarrow \infty$ by definition of convergence in $\mathscr{S}$, hence $\|\xi\|^{\mathbf{m}} D^{\mathbf{p}}(\mathscr{F}[\varphi_j]) \rightarrow 0$ uniformly, so that $(\mathscr{F}[\varphi_j]) \rightarrow 0$ in $\mathscr{S}$ as $j \rightarrow \infty$. The same proof applies to $\bar{\mathscr{F}}$.

1.3.2.5.3. Definition and examples of tempered distributions

A distribution $T \in \mathscr{D}'(\mathbb{R}^n)$ is said to be *tempered* if it can be extended into a continuous linear functional on $\mathscr{S}$.

If $\mathscr{S}'(\mathbb{R}^n)$ is the topological dual of $\mathscr{S}(\mathbb{R}^n)$, and if $S \in \mathscr{S}'(\mathbb{R}^n)$, then its restriction to $\mathscr{D}$ is a tempered distribution; conversely, if $T \in \mathscr{D}'$ is tempered, then its extension to $\mathscr{S}$ is unique (because $\mathscr{D}$ is dense in $\mathscr{S}$), hence it defines an element S of $\mathscr{S}'$. We may therefore identify $\mathscr{S}'$ and the space of tempered distributions.

A distribution with compact support is tempered, *i.e.* $\mathscr{S}' \supset \mathscr{E}'$. By transposition of the corresponding properties of $\mathscr{S}$, it is readily established that the derivative, translate or product by a polynomial of a tempered distribution is still a tempered distribution.

These inclusion relations may be summarized as follows: since $\mathscr{S}$ contains $\mathscr{D}$ but is contained in $\mathscr{E}$, the reverse inclusions hold for the topological duals, and hence $\mathscr{S}'$ contains $\mathscr{E}'$ but is contained in $\mathscr{D}'$.

A locally summable function f on $\mathbb{R}^n$ will be said to be *of polynomial growth* if $|f(\mathbf{x})|$ can be majorized by a polynomial in $\|\mathbf{x}\|$ as $\|\mathbf{x}\| \rightarrow \infty$. It is easily shown that such a function f defines a tempered distribution T_f *via*

$$\langle T_f, \varphi \rangle = \int_{\mathbb{R}^n} f(\mathbf{x})\varphi(\mathbf{x}) \,\mathrm{d}^n\mathbf{x}.$$

In particular, polynomials over $\mathbb{R}^n$ define tempered distributions, and so do functions in $\mathscr{S}$. The latter remark, together with the transposition identity (Section 1.3.2.4.4), invites the extension of $\mathscr{F}$ and $\bar{\mathscr{F}}$ from $\mathscr{S}$ to $\mathscr{S}'$.

1.3.2.5.4. Fourier transforms of tempered distributions

The Fourier transform $\mathscr{F}[T]$ and cotransform $\bar{\mathscr{F}}[T]$ of a tempered distribution T are defined by

$$\langle \mathscr{F}[T], \varphi \rangle = \langle T, \mathscr{F}[\varphi] \rangle$$
$$\langle \bar{\mathscr{F}}[T], \varphi \rangle = \langle T, \bar{\mathscr{F}}[\varphi] \rangle$$

for all test functions $\varphi \in \mathscr{S}$. Both $\mathscr{F}[T]$ and $\bar{\mathscr{F}}[T]$ are themselves tempered distributions, since the maps $\varphi \longmapsto \mathscr{F}[\varphi]$ and $\varphi \longmapsto \bar{\mathscr{F}}[\varphi]$ are both linear and continuous for the topology of $\mathscr{S}$. In the same way that $\mathbf{x}$ and $\boldsymbol{\xi}$ have been used consistently as arguments for φ and $\mathscr{F}[\varphi]$, respectively, the notation $T_{\mathbf{x}}$ and $\mathscr{F}[T]_{\boldsymbol{\xi}}$ will be used to indicate which variables are involved.

When T is a distribution with compact support, its Fourier transform may be written

$$\mathscr{F}[T_{\mathbf{x}}]_{\boldsymbol{\xi}} = \langle T_{\mathbf{x}}, \exp(-2\pi i \boldsymbol{\xi} \cdot \mathbf{x}) \rangle$$

since the function $\mathbf{x} \longmapsto \exp(-2\pi i \boldsymbol{\xi} \cdot \mathbf{x})$ is in $\mathscr{E}$ while $T_{\mathbf{x}} \in \mathscr{E}'$. It can be shown, as in Section 1.3.2.4.2, to be analytically continuable into an entire function over $\mathbb{C}^n$.

1.3.2.5.5. Transposition of basic properties

The duality between differentiation and multiplication by a monomial extends from $\mathscr{S}$ to $\mathscr{S}'$ by transposition:

$$\mathscr{F}[D_{\mathbf{x}}^{\mathbf{p}} T_{\mathbf{x}}]_{\boldsymbol{\xi}} = (2\pi i \boldsymbol{\xi})^{\mathbf{p}} \mathscr{F}[T_{\mathbf{x}}]_{\boldsymbol{\xi}}$$
$$D_{\boldsymbol{\xi}}^{\mathbf{p}}(\mathscr{F}[T_{\mathbf{x}}]_{\boldsymbol{\xi}}) = \mathscr{F}[(-2\pi i \mathbf{x})^{\mathbf{p}} T_{\mathbf{x}}]_{\boldsymbol{\xi}}.$$

Analogous formulae hold for $\bar{\mathscr{F}}$, with i replaced by $-i$.

The formulae expressing the duality between translation and phase shift, *e.g.*

$$\mathscr{F}[\tau_{\mathbf{a}} T_{\mathbf{x}}]_{\boldsymbol{\xi}} = \exp(-2\pi i \mathbf{a} \cdot \boldsymbol{\xi}) \mathscr{F}[T_{\mathbf{x}}]_{\boldsymbol{\xi}}$$
$$\tau_{\boldsymbol{\alpha}}(\mathscr{F}[T_{\mathbf{x}}]_{\boldsymbol{\xi}}) = \mathscr{F}[\exp(2\pi i \boldsymbol{\alpha} \cdot \mathbf{x}) T_{\mathbf{x}}]_{\boldsymbol{\xi}};$$

between a linear change of variable and its contragredient, *e.g.*

$$\mathscr{F}[A^{\#} T] = |\det \mathbf{A}| [(\mathbf{A}^{-1})^T]^{\#} \mathscr{F}[T];$$

are obtained similarly by transposition from the corresponding identities in $\mathscr{S}$. They give a transposition formula for an affine change of variables $\mathbf{x} \longmapsto S(\mathbf{x}) = \mathbf{A}\mathbf{x} + \mathbf{b}$ with nonsingular matrix $\mathbf{A}$:

$$\begin{aligned} \mathscr{F}[S^{\#} T] &= \exp(-2\pi i \boldsymbol{\xi} \cdot \mathbf{b}) \mathscr{F}[A^{\#} T] \\ &= \exp(-2\pi i \boldsymbol{\xi} \cdot \mathbf{b}) |\det \mathbf{A}| [(\mathbf{A}^{-1})^T]^{\#} \mathscr{F}[T], \end{aligned}$$

with a similar result for $\bar{\mathscr{F}}$, replacing $-i$ by $+i$.

Conjugate symmetry is obtained similarly:

$$\mathscr{F}[\bar{T}] = \overline{\breve{\mathscr{F}[T]}}, \quad \mathscr{F}[\breve{\bar{T}}] = \overline{\mathscr{F}[T]},$$

with the same identities for $\bar{\mathscr{F}}$.

The tensor product property also transposes to tempered distributions: if $U \in \mathscr{S}'(\mathbb{R}^m)$, $V \in \mathscr{S}'(\mathbb{R}^n)$,

$$\mathscr{F}[U_{\mathbf{x}} \otimes V_{\mathbf{y}}] = \mathscr{F}[U]_{\boldsymbol{\xi}} \otimes \mathscr{F}[V]_{\boldsymbol{\eta}}$$
$$\bar{\mathscr{F}}[U_{\mathbf{x}} \otimes V_{\mathbf{y}}] = \bar{\mathscr{F}}[U]_{\boldsymbol{\xi}} \otimes \bar{\mathscr{F}}[V]_{\boldsymbol{\eta}}.$$

1.3.2.5.6. Transforms of δ-functions

Since δ has compact support,

$$\mathscr{F}[\delta_{\mathbf{x}}]_{\boldsymbol{\xi}} = \langle \delta_{\mathbf{x}}, \exp(-2\pi i \boldsymbol{\xi} \cdot \mathbf{x}) \rangle = 1_{\boldsymbol{\xi}}, \quad i.e. \ \mathscr{F}[\delta] = 1.$$

It is instructive to show that conversely $\mathscr{F}[1] = \delta$ without invoking the reciprocity theorem. Since $\partial_j 1 = 0$ for all $j = 1, \ldots, n$, it follows from Section 1.3.2.3.9.4 that $\mathscr{F}[1] = c\delta$; the

references