International Tables for Crystallography (2010). Vol. B, Section 1.3.2.7.2.4, pp. 48–49.

1.3.2.7.2. Duality between subdivision and decimation of period lattices

1.3.2.7.2.1. Geometric description of sublattices

Let $\Lambda_{\mathbf{A}}$ be a period lattice in $\mathbb{R}^n$ with matrix $\mathbf{A}$, and let $\Lambda_{\mathbf{A}}^*$ be the lattice reciprocal to $\Lambda_{\mathbf{A}}$, with period matrix $(A^{-1})^T$. Let $\Lambda_{\mathbf{B}}, \mathbf{B}, \Lambda_{\mathbf{B}}^*$ be defined similarly, and let us suppose that $\Lambda_{\mathbf{A}}$ *is a sublattice of* $\Lambda_{\mathbf{B}}$, *i.e.* that $\Lambda_{\mathbf{B}} \supset \Lambda_{\mathbf{A}}$ as a set.

The relation between $\Lambda_{\mathbf{A}}$ and $\Lambda_{\mathbf{B}}$ may be described in two different fashions: (i) multiplicatively, and (ii) additively.

(i) We may write $\mathbf{A} = \mathbf{BN}$ for some nonsingular matrix $\mathbf{N}$ with integer entries. $\mathbf{N}$ may be viewed as the period matrix of the coarser lattice $\Lambda_{\mathbf{A}}$ with respect to the period basis of the finer lattice $\Lambda_{\mathbf{B}}$. It will be more convenient to write $\mathbf{A} = \mathbf{DB}$, where $\mathbf{D} = \mathbf{BNB}^{-1}$ is a rational matrix (with integer determinant since $\det \mathbf{D} = \det \mathbf{N}$) in terms of which the two lattices are related by

$$\Lambda_{\mathbf{A}} = \mathbf{D}\Lambda_{\mathbf{B}}.$$

(ii) Call two vectors in $\Lambda_{\mathbf{B}}$ *congruent modulo* $\Lambda_{\mathbf{A}}$ if their difference lies in $\Lambda_{\mathbf{A}}$. Denote the set of congruence classes (or 'cosets') by $\Lambda_{\mathbf{B}}/\Lambda_{\mathbf{A}}$, and the number of these classes by $[\Lambda_{\mathbf{B}} : \Lambda_{\mathbf{A}}]$. The 'coset decomposition'

$$\Lambda_{\mathbf{B}} = \bigcup_{\ell \in \Lambda_{\mathbf{B}}/\Lambda_{\mathbf{A}}} (\ell + \Lambda_{\mathbf{A}})$$

represents $\Lambda_{\mathbf{B}}$ as the disjoint union of $[\Lambda_{\mathbf{B}} : \Lambda_{\mathbf{A}}]$ translates of $\Lambda_{\mathbf{A}}$. $\Lambda_{\mathbf{B}}/\Lambda_{\mathbf{A}}$ is a finite lattice with $[\Lambda_{\mathbf{B}} : \Lambda_{\mathbf{A}}]$ elements, called the *residual lattice* of $\Lambda_{\mathbf{B}}$ modulo $\Lambda_{\mathbf{A}}$.

The two descriptions are connected by the relation $[\Lambda_{\mathbf{B}} : \Lambda_{\mathbf{A}}] = \det \mathbf{D} = \det \mathbf{N}$, which follows from a volume calculation. We may also combine (i) and (ii) into

$$\text{(iii)} \qquad \Lambda_{\mathbf{B}} = \bigcup_{\ell \in \Lambda_{\mathbf{B}}/\Lambda_{\mathbf{A}}} (\ell + \mathbf{D}\Lambda_{\mathbf{B}})$$

which may be viewed as the n-dimensional equivalent of the Euclidean algorithm for integer division: ℓ is the 'remainder' of the division by $\Lambda_{\mathbf{A}}$ of a vector in $\Lambda_{\mathbf{B}}$, the quotient being the matrix $\mathbf{D}$.

1.3.2.7.2.2. Sublattice relations for reciprocal lattices

Let us now consider the two *reciprocal lattices* $\Lambda_{\mathbf{A}}^*$ and $\Lambda_{\mathbf{B}}^*$. Their period matrices $(\mathbf{A}^{-1})^T$ and $(\mathbf{B}^{-1})^T$ are related by: $(\mathbf{B}^{-1})^T = (\mathbf{A}^{-1})^T\mathbf{N}^T$, where $\mathbf{N}^T$ is an integer matrix; or equivalently by $(\mathbf{B}^{-1})^T = \mathbf{D}^T(\mathbf{A}^{-1})^T$. This shows that the roles are reversed in that $\Lambda_{\mathbf{B}}^*$ *is a sublattice of* $\Lambda_{\mathbf{A}}^*$, which we may write:

$$\text{(i)}^* \qquad \Lambda_{\mathbf{B}}^* = \mathbf{D}^T\Lambda_{\mathbf{A}}^*$$

$$\text{(ii)}^* \qquad \Lambda_{\mathbf{A}}^* = \bigcup_{\ell^* \in \Lambda_{\mathbf{A}}^*/\Lambda_{\mathbf{B}}^*} (\ell^* + \Lambda_{\mathbf{B}}^*).$$

The residual lattice $\Lambda_{\mathbf{A}}^*/\Lambda_{\mathbf{B}}^*$ is finite, with $[\Lambda_{\mathbf{A}}^* : \Lambda_{\mathbf{B}}^*] = \det \mathbf{D} = \det \mathbf{N} = [\Lambda_{\mathbf{B}} : \Lambda_{\mathbf{A}}]$, and we may again combine (i)* and (ii)* into

$$\text{(iii)}^* \qquad \Lambda_{\mathbf{A}}^* = \bigcup_{\ell^* \in \Lambda_{\mathbf{A}}^*/\Lambda_{\mathbf{B}}^*} (\ell^* + \mathbf{D}^T\Lambda_{\mathbf{A}}^*).$$

1.3.2.7.2.3. Relation between lattice distributions

The above relations between lattices may be rewritten in terms of the corresponding *lattice distributions* as follows:

$$\text{(i)} \qquad R_{\mathbf{A}} = \frac{1}{|\det \mathbf{D}|}\mathbf{D}^{\#}R_{\mathbf{B}}^*$$

$$\text{(ii)} \qquad R_{\mathbf{B}} = T_{\mathbf{B}/\mathbf{A}} * R_{\mathbf{A}}$$

$$\text{(i)}^* \qquad R_{\mathbf{B}}^* = \frac{1}{|\det \mathbf{D}|}(\mathbf{D}^T)^{\#}R_{\mathbf{A}}^*$$

$$\text{(ii)}^* \qquad R_{\mathbf{A}}^* = T_{\mathbf{A}/\mathbf{B}}^* * R_{\mathbf{B}}^*$$

where

$$T_{\mathbf{B}/\mathbf{A}} = \sum_{\ell \in \Lambda_{\mathbf{B}}/\Lambda_{\mathbf{A}}} \delta_{(\ell)}$$

and

$$T_{\mathbf{A}/\mathbf{B}}^* = \sum_{\ell^* \in \Lambda_{\mathbf{A}}^*/\Lambda_{\mathbf{B}}^*} \delta_{(\ell^*)}$$

are (finite) residual-lattice distributions. We may incorporate the factor $1/|\det \mathbf{D}|$ in (i) and (i)* into these distributions and define

$$S_{\mathbf{B}/\mathbf{A}} = \frac{1}{|\det \mathbf{D}|}T_{\mathbf{B}/\mathbf{A}}, \quad S_{\mathbf{A}/\mathbf{B}}^* = \frac{1}{|\det \mathbf{D}|}T_{\mathbf{A}/\mathbf{B}}^*.$$

Since $|\det \mathbf{D}| = [\Lambda_{\mathbf{B}} : \Lambda_{\mathbf{A}}] = [\Lambda_{\mathbf{A}}^* : \Lambda_{\mathbf{B}}^*]$, convolution with $S_{\mathbf{B}/\mathbf{A}}$ and $S_{\mathbf{A}/\mathbf{B}}^*$ has the effect of *averaging* the translates of a distribution under the elements (or 'cosets') of the residual lattices $\Lambda_{\mathbf{B}}/\Lambda_{\mathbf{A}}$ and $\Lambda_{\mathbf{A}}^*/\Lambda_{\mathbf{B}}^*$, respectively. This process will be called 'coset averaging'. Eliminating $R_{\mathbf{A}}$ and $R_{\mathbf{B}}$ between (i) and (ii), and $R_{\mathbf{A}}^*$ and $R_{\mathbf{B}}^*$ between (i)* and (ii)*, we may write:

$$\text{(i}'\text{)} \qquad R_{\mathbf{A}} = \mathbf{D}^{\#}(S_{\mathbf{B}/\mathbf{A}} * R_{\mathbf{A}})$$

$$\text{(ii}'\text{)} \qquad R_{\mathbf{B}} = S_{\mathbf{B}/\mathbf{A}} * (\mathbf{D}^{\#}R_{\mathbf{B}})$$

$$\text{(i}'\text{)}^* \qquad R_{\mathbf{B}}^* = (\mathbf{D}^T)^{\#}(S_{\mathbf{A}/\mathbf{B}}^* * R_{\mathbf{B}}^*)$$

$$\text{(ii}'\text{)}^* \qquad R_{\mathbf{A}}^* = S_{\mathbf{A}/\mathbf{B}}^* * [(\mathbf{D}^T)^{\#}R_{\mathbf{A}}^*].$$

These identities show that *period subdivision* by convolution with $S_{\mathbf{B}/\mathbf{A}}$ (respectively $S_{\mathbf{A}/\mathbf{B}}^*$) on the one hand, and *period decimation* by 'dilation' by $\mathbf{D}^{\#}$ on the other hand, are mutually inverse operations on $R_{\mathbf{A}}$ and $R_{\mathbf{B}}$ (respectively $R_{\mathbf{A}}^*$ and $R_{\mathbf{B}}^*$).

1.3.2.7.2.4. Relation between Fourier transforms

Finally, let us consider the relations between the *Fourier transforms* of these lattice distributions. Recalling the basic relation of Section 1.3.2.6.5,

$$\begin{aligned}
\mathscr{F}[R_{\mathbf{A}}] &= \frac{1}{|\det \mathbf{A}|}R_{\mathbf{A}}^* \\
&= \frac{1}{|\det \mathbf{DB}|}T_{\mathbf{A}/\mathbf{B}}^* * R_{\mathbf{B}}^* \qquad \text{by (ii)}^* \\
&= \left(\frac{1}{|\det \mathbf{D}|}T_{\mathbf{A}/\mathbf{B}}^*\right) * \left(\frac{1}{|\det \mathbf{B}|}R_{\mathbf{B}}^*\right)
\end{aligned}$$

i.e.

$$\text{(iv)} \qquad \mathcal{F}[R_{\mathbf{A}}] = S^*_{\mathbf{A}/\mathbf{B}} * \mathcal{F}[R_{\mathbf{B}}]$$

and similarly:

$$\text{(v)} \qquad \mathcal{F}[R^*_{\mathbf{B}}] = S_{\mathbf{B}/\mathbf{A}} * \mathcal{F}[R^*_{\mathbf{A}}].$$

Thus $R_{\mathbf{A}}$ (respectively $R^*_{\mathbf{B}}$), a *decimated* version of $R_{\mathbf{B}}$ (respectively $R^*_{\mathbf{A}}$), is transformed by $\mathcal{F}$ into a *subdivided* version of $\mathcal{F}[R_{\mathbf{B}}]$ (respectively $\mathcal{F}[R^*_{\mathbf{A}}]$).

The converse is also true:

$$\begin{aligned}\mathcal{F}[R_{\mathbf{B}}] &= \frac{1}{|\det \mathbf{B}|} R^*_{\mathbf{B}} \\ &= \frac{1}{|\det \mathbf{B}|}\frac{1}{|\det \mathbf{D}|}(\mathbf{D}^T)^{\#} R^*_{\mathbf{A}} \qquad \text{by (i)}^* \\ &= (\mathbf{D}^T)^{\#}\left(\frac{1}{|\det \mathbf{A}|} R^*_{\mathbf{A}}\right)\end{aligned}$$

i.e.

$$\text{(iv}') \qquad \mathcal{F}[R_{\mathbf{B}}] = (\mathbf{D}^T)^{\#}\mathcal{F}[R_{\mathbf{A}}]$$

and similarly

$$\text{(v}') \qquad \mathcal{F}[R^*_{\mathbf{A}}] = \mathbf{D}^{\#}\mathcal{F}[R^*_{\mathbf{B}}].$$

Thus $R_{\mathbf{B}}$ (respectively $R^*_{\mathbf{A}}$), a *subdivided* version of $R_{\mathbf{A}}$ (respectively $R^*_{\mathbf{B}}$) is transformed by $\mathcal{F}$ into a *decimated* version of $\mathcal{F}[R_{\mathbf{A}}]$ (respectively $\mathcal{F}[R^*_{\mathbf{B}}]$). Therefore, *the Fourier transform exchanges subdivision and decimation of period lattices for lattice distributions*.

Further insight into this phenomenon is provided by applying $\bar{\mathcal{F}}$ to both sides of (iv) and (v) and invoking the convolution theorem:

$$\begin{aligned}&\text{(iv}'') \qquad R_{\mathbf{A}} = \bar{\mathcal{F}}[S^*_{\mathbf{A}/\mathbf{B}}] \times R_{\mathbf{B}} \\ &\text{(v}'') \qquad R^*_{\mathbf{B}} = \bar{\mathcal{F}}[S_{\mathbf{B}/\mathbf{A}}] \times R^*_{\mathbf{A}}.\end{aligned}$$

These identities show that multiplication by the transform of the period-subdividing distribution $S^*_{\mathbf{A}/\mathbf{B}}$ (respectively $S_{\mathbf{B}/\mathbf{A}}$) has the effect of decimating $R_{\mathbf{B}}$ to $R_{\mathbf{A}}$ (respectively $R^*_{\mathbf{A}}$ to $R^*_{\mathbf{B}}$). They clearly imply that, if $\ell \in \Lambda_{\mathbf{B}}/\Lambda_{\mathbf{A}}$ and $\ell^* \in \Lambda^*_{\mathbf{A}}/\Lambda^*_{\mathbf{B}}$, then

$$\begin{aligned}\bar{\mathcal{F}}[S^*_{\mathbf{A}/\mathbf{B}}](\ell) &= 1 \text{ if } \ell = \mathbf{0} \quad (\textit{i.e.}\text{ if } \ell \text{ belongs to the class of } \Lambda_{\mathbf{A}}), \\ &= 0 \text{ if } \ell \neq \mathbf{0}; \\ \bar{\mathcal{F}}[S_{\mathbf{B}/\mathbf{A}}](\ell^*) &= 1 \text{ if } \ell^* = \mathbf{0} \quad (\textit{i.e.}\text{ if } \ell^* \text{ belongs to the class of } \Lambda^*_{\mathbf{B}}), \\ &= 0 \text{ if } \ell^* \neq \mathbf{0}.\end{aligned}$$

Therefore, the duality between subdivision and decimation may be viewed as another aspect of that between convolution and multiplication.

There is clearly a strong analogy between the sampling/periodization duality of Section 1.3.2.6.6 and the decimation/subdivision duality, which is viewed most naturally in terms of subgroup relationships: both sampling and decimation involve restricting a function to a *discrete additive subgroup* of the domain over which it is initially given.

1.3.2.7.2.5. Sublattice relations in terms of periodic distributions

The usual presentation of this duality is not in terms of lattice distributions, but of periodic distributions obtained by convolving them with a motif.

Given $T^0 \in \mathcal{E}'(\mathbb{R}^n)$, let us form $R_{\mathbf{A}} * T^0$, then *decimate* its transform $(1/|\det \mathbf{A}|)R^*_{\mathbf{A}} \times \bar{\mathcal{F}}[T^0]$ by keeping only its values at the points of the coarser lattice $\Lambda^*_{\mathbf{B}} = \mathbf{D}^T\Lambda^*_{\mathbf{A}}$; as a result, $R^*_{\mathbf{A}}$ is replaced by $(1/|\det \mathbf{D}|)R^*_{\mathbf{B}}$, and the reverse transform then yields

$$\frac{1}{|\det \mathbf{D}|} R_{\mathbf{B}} * T^0 = S_{\mathbf{B}/\mathbf{A}} * (R_{\mathbf{A}} * T^0) \qquad \text{by (ii)},$$

which is the *coset-averaged* version of the original $R_{\mathbf{A}} * T^0$. The converse situation is analogous to that of Shannon's sampling theorem. Let a function $\varphi \in \mathcal{E}(\mathbb{R}^n)$ whose transform $\Phi = \mathcal{F}[\varphi]$ has compact support be sampled as $R_{\mathbf{B}} \times \varphi$ at the nodes of $\Lambda_{\mathbf{B}}$. Then

$$\mathcal{F}[R_{\mathbf{B}} \times \varphi] = \frac{1}{|\det \mathbf{B}|}(R^*_{\mathbf{B}} * \Phi)$$

is periodic with period lattice $\Lambda^*_{\mathbf{B}}$. If the sampling lattice $\Lambda_{\mathbf{B}}$ is decimated to $\Lambda_{\mathbf{A}} = \mathbf{D}\Lambda_{\mathbf{B}}$, the inverse transform becomes

$$\begin{aligned}\mathcal{F}[R_{\mathbf{A}} \times \varphi] &= \frac{1}{|\det \mathbf{D}|}(R^*_{\mathbf{A}} * \Phi) \\ &= S^*_{\mathbf{A}/\mathbf{B}} * (R^*_{\mathbf{B}} * \Phi) \qquad \text{by (ii)}^*,\end{aligned}$$

hence becomes periodized more finely by averaging over the cosets of $\Lambda^*_{\mathbf{A}}/\Lambda^*_{\mathbf{B}}$. With this finer periodization, the various copies of Supp Φ may start to overlap (a phenomenon called 'aliasing'), indicating that decimation has produced too coarse a sampling of φ.

1.3.2.7.3. Discretization of the Fourier transformation

Let $\varphi^0 \in \mathcal{E}(\mathbb{R}^n)$ be such that $\Phi^0 = \mathcal{F}[\varphi^0]$ has compact support (φ^0 is said to be *band-limited*). Then $\varphi = R_{\mathbf{A}} * \varphi^0$ is $\Lambda_{\mathbf{A}}$-periodic, and $\Phi = \mathcal{F}[\varphi] = (1/|\det \mathbf{A}|)R^*_{\mathbf{A}} \times \Phi^0$ is such that only a finite number of points $\lambda^*_{\mathbf{A}}$ of $\Lambda^*_{\mathbf{A}}$ have a nonzero Fourier coefficient $\Phi^0(\lambda^*_{\mathbf{A}})$ attached to them. We may therefore find a *decimation* $\Lambda^*_{\mathbf{B}} = \mathbf{D}^T\Lambda^*_{\mathbf{A}}$ of $\Lambda^*_{\mathbf{A}}$ such that the distinct translates of Supp Φ^0 by vectors of $\Lambda^*_{\mathbf{B}}$ do not intersect.

The distribution Φ can be uniquely recovered from $R^*_{\mathbf{B}} * \Phi$ by the procedure of Section 1.3.2.7.1, and we may write:

$$\begin{aligned}R^*_{\mathbf{B}} * \Phi &= \frac{1}{|\det \mathbf{A}|} R^*_{\mathbf{B}} * (R^*_{\mathbf{A}} \times \Phi^0) \\ &= \frac{1}{|\det \mathbf{A}|} R^*_{\mathbf{A}} \times (R^*_{\mathbf{B}} * \Phi^0) \\ &= \frac{1}{|\det \mathbf{A}|} R^*_{\mathbf{B}} * [T^*_{\mathbf{A}/\mathbf{B}} \times (R^*_{\mathbf{B}} * \Phi^0)];\end{aligned}$$

these rearrangements being legitimate because Φ^0 and $T^*_{\mathbf{A}/\mathbf{B}}$ have compact supports which are intersection-free under the action of $\Lambda^*_{\mathbf{B}}$. By virtue of its $\Lambda^*_{\mathbf{B}}$-periodicity, this distribution is entirely characterized by its 'motif' $\tilde{\Phi}$ with respect to $\Lambda^*_{\mathbf{B}}$:

$$\tilde{\Phi} = \frac{1}{|\det \mathbf{A}|} T^*_{\mathbf{A}/\mathbf{B}} \times (R^*_{\mathbf{B}} * \Phi^0).$$

references