This is only an approximation, because the residuals $1 - \mathsf{b}^T\mathsf{r}$ are not the differences between the observations and appropriate linear functions of the parameters, nor are their variances (the $\mathsf{b}^T\mathbf{M}\mathsf{b}$'s) independent of the parameters (or, in turn, the errors in the observations).

We ask also about the perpendicular distances, e, of atoms to plane and the mean-square deviation $\overline{(\delta e)^2}$ to be expected in e.

$$e = (1 - \mathsf{b}^T\mathsf{r})/\sqrt{\mathsf{b}^T\mathsf{b}} = d(1 - \mathsf{b}^T\mathsf{r})$$
$$\delta e = -d(\mathsf{b}^T\eta + \mathsf{r}^T\varepsilon) + O(\eta^2).$$

Here η and ε are the errors in r and b. Neglecting '$O(\eta^2)$' then leads to

$$\overline{(\delta e)^2} = d^2(\mathsf{b}^T\overline{\eta\eta^T}\mathsf{b} + 2\mathsf{b}^T\overline{\eta\varepsilon^T}\mathsf{r} + \mathsf{r}^T\overline{\varepsilon\varepsilon^T}\mathsf{r}).$$

We have $\overline{\eta\eta^T} = \mathbf{M} = \mathbf{P}^{-1}$, but $\overline{\eta\varepsilon^T}$ and $\overline{\varepsilon\varepsilon^T}$ perhaps still need to be evaluated.

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