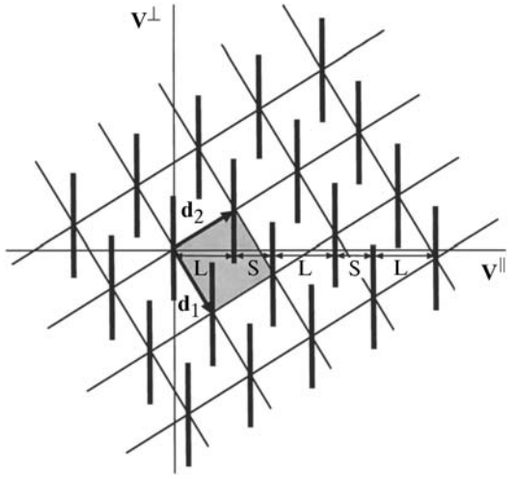


Fig. 4.6.2.8. 2D embedding of the Fibonacci chain. The short and long distances S and L, generated by the intersection of the atomic surfaces with the physical space $\mathbf{V}^\parallel$, are indicated. The atomic surfaces are represented by bars parallel to $\mathbf{V}^\perp$. Their lengths correspond to the projection of one unit cell (shaded) upon $\mathbf{V}^\perp$.

$$\mathbf{r} = n_1\mathbf{d}_1 + n_2\mathbf{d}_2, \text{ with } \mathbf{d}_1 = \frac{1}{a^*(2+\tau)}\begin{pmatrix}1\\-\tau\end{pmatrix}_V,$$

$$\mathbf{d}_2 = \frac{1}{a^*(2+\tau)}\begin{pmatrix}\tau\\1\end{pmatrix}_V;$$

$$\mathbf{H} = h_1\mathbf{d}_1^* + h_2\mathbf{d}_2^*, \text{ with } \mathbf{d}_1^* = a^*\begin{pmatrix}1\\-\tau\end{pmatrix}_V, \mathbf{d}_2^* = a^*\begin{pmatrix}\tau\\1\end{pmatrix}_V.$$

The 1D Fibonacci chain results from the cut of the parallel (physical) space with the 2D lattice $\Sigma$ decorated with line elements for the *atomic surfaces* (*acceptance domains*). In this description, the atomic surfaces correspond simply to the projection of one 2D unit cell upon the perpendicular-space coordinate. This satisfies the condition that each unit cell contributes exactly to one point of the Fibonacci chain (*primitive unit cell*). The physical space $\mathbf{V}^\parallel$ is related to the eigenspace of the substitution matrix $S$ associated with its eigenvalue $\lambda_1 = \tau$. The perpendicular space $\mathbf{V}^\perp$ corresponds to the eigenspace of the substitution matrix $S$ associated with its eigenvalue $\lambda_2 = -1/\tau$. Thus, the physical space scales to powers of $\tau$ and the perpendicular space to powers of $-1/\tau$.

By block-diagonalization, the reducible substitution (scaling) matrix $S$ can be decomposed into two non-equivalent irreducible representations. These can be assigned to the two 1D orthogonal subspaces $\mathbf{V}^\parallel$ and $\mathbf{V}^\perp$ forming the 2D embedding space $\mathbf{V} = \mathbf{V}^\parallel \oplus \mathbf{V}^\perp$. Thus, using $WSW^{-1} = S_V = S_V^\parallel \oplus S_V^\perp$, where

$$W = \begin{pmatrix}1 & \tau\\ -\tau & 1\end{pmatrix} = \begin{pmatrix}\mathbf{d}_1^* & \mathbf{d}_2^*\end{pmatrix},$$

one obtains

$$\begin{pmatrix}1 & \tau\\ -\tau & 1\end{pmatrix}\begin{pmatrix}0 & 1\\ 1 & 1\end{pmatrix}_D\begin{pmatrix}1 & \tau\\ -\tau & 1\end{pmatrix}^{-1} = \left(\begin{array}{c|c}\tau & 0\\ \hline 0 & -1/\tau\end{array}\right)_V = \left(\begin{array}{c|c}S^\parallel & 0\\ \hline 0 & S^\perp\end{array}\right)_V,$$

the scaling operations $S^\parallel$ and $S^\perp$ in parallel and in perpendicular space as indicated by the partition lines.

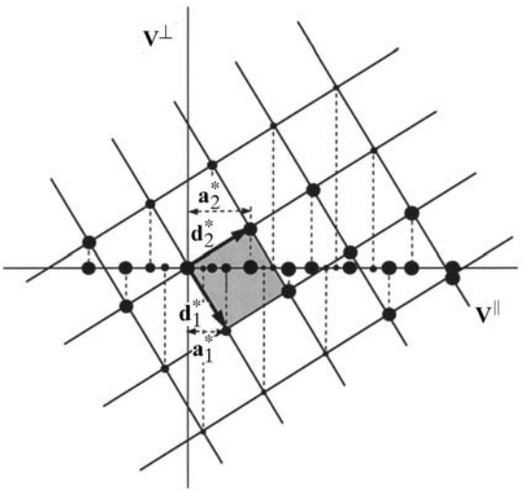


Fig. 4.6.2.9. Schematic representation of the reciprocal space of the embedded Fibonacci chain depicted in Fig. 4.6.2.8. The physical-space reciprocal basis $\mathbf{a}_1^*$ and $\mathbf{a}_2^*$ is marked. The diameters of the filled circles are roughly proportional to the reflection intensities. One 2D reciprocal-lattice unit cell is shadowed. The actual 1D diffraction pattern of the 1D Fibonacci chain results from a projection of the 2D reciprocal space onto the parallel space. The correspondence between 2D reciprocal-lattice positions and their projected images is indicated by dashed lines.

The metric tensors for the reciprocal and the direct 2D square lattices read

$$G^* = |a^*|^2(2+\tau)\begin{pmatrix}1 & 0\\ 0 & 1\end{pmatrix} \text{ and } G = \frac{1}{|a^*|^2(2+\tau)}\begin{pmatrix}1 & 0\\ 0 & 1\end{pmatrix}.$$

The short distance S of the Fibonacci sequence is related to $a^*$ by

$$\begin{aligned} S &= 1/[a^*(2+\tau)]\\ &= \min\left\{\left|\pi^\parallel(\mathbf{d}_i - \mathbf{d}_j)\right| \,\middle|\, \left|\pi^\perp(\mathbf{d}_i - \mathbf{d}_j)\right| < \Omega_{\rm AS} \wedge i, j \in \mathbb{Z}\right\}, \end{aligned}$$

with the projectors $\pi^\parallel$ and $\pi^\perp$ onto $\mathbf{V}^\parallel$ and $\mathbf{V}^\perp$. The *point density* $\rho_{\rm p}$ of the Fibonacci chain, *i.e.* the number of vertices per unit length, can be calculated using the formula

$$\rho_{\rm p} = \frac{\Omega_{\rm AS}}{\Omega_{\rm UC}} = \frac{(1+\tau)/[a^*(2+\tau)]}{1/[|a^*|^2(2+\tau)]} = a^*\tau^2,$$

where $\Omega_{\rm AS}$ and $\Omega_{\rm UC}$ are the areas of the atomic surface and of the 2D unit cell, respectively.

For an infinite Fibonacci sequence generated from the intervals S and L an average distance $d$ can be calculated:

$$d = \lim_{n\to\infty}\frac{F_n\mathrm{S} + F_{n+1}\mathrm{L}}{F_n + F_{n+1}} = \lim_{n\to\infty}\frac{F_n(\mathrm{S}+\tau\mathrm{L})}{F_n(1+\tau)} = \frac{\mathrm{S}(1+\tau^2)}{(1+\tau)} = (3-\tau)\mathrm{S}.$$

Therefrom, the point density can also be calculated:

$$\rho_{\rm p} = 1/d = 1/[(3-\tau)\mathrm{S}] = [a^*(2+\tau)]/(3-\tau) = a^*\tau^2.$$

An approximant structure of the Fibonacci sequence with a unit cell containing $m$ intervals L and $n$ intervals S can be generated by shearing the 2D lattice $\Sigma$ by the shear matrix $S_m$,