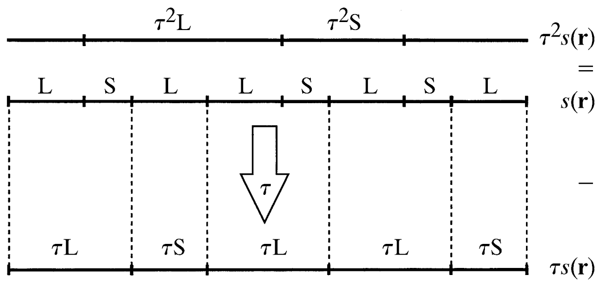

Fig. 4.6.3.10. Part … LSLLSLSL … of a Fibonacci sequence $s(\mathbf{r})$ before and after scaling by the factor τ. L is mapped onto τL, S onto τS = L. The vertices of the new sequence are a subset of those of the original sequence (the correspondence is indicated by dashed lines). The residual vertices $\tau^2 s(\mathbf{r})$, which give when decorating $\tau s(\mathbf{r})$ the Fibonacci sequence $s(\mathbf{r})$, form a Fibonacci sequence scaled by a factor τ^2.

$$F(\mathbf{H}) = F(\tau\mathbf{H}) + \exp(2\pi i\tau\mathbf{H})F(\tau^2\mathbf{H}).$$

Hence, phases of structure factors that are related by scaling symmetry can be determined from each other.

Further scaling relationships in reciprocal space exist: scaling a diffraction vector

$$\mathbf{H} = h_1\mathbf{d}_1^* + h_2\mathbf{d}_2^* = h_1 a^* \begin{pmatrix} 1 \\ -\tau \end{pmatrix}_V + h_2 a^* \begin{pmatrix} \tau \\ 1 \end{pmatrix}_V$$

with the matrix

$$S = \begin{pmatrix} 0 & 1 \\ 1 & 1 \end{pmatrix}_D,$$

$$\begin{pmatrix} 0 & 1 \\ 1 & 1 \end{pmatrix}_D \begin{pmatrix} h_1 \\ h_2 \end{pmatrix}_D = \begin{pmatrix} F_n & F_{n+1} \\ F_{n+1} & F_{n+2} \end{pmatrix}_D \begin{pmatrix} h_1 \\ h_2 \end{pmatrix}_D = \begin{pmatrix} F_n h_1 + F_{n+1} h_2 \\ F_{n+1} h_1 + F_{n+2} h_2 \end{pmatrix}_D,$$

increases the magnitudes of structure factors assigned to this particular diffraction vector $\mathbf{H}$,

$$|F(S^n\mathbf{H})| > |F(S^{n-1}\mathbf{H})| > \ldots > |F(S\mathbf{H})| > |F(\mathbf{H})|.$$

This is due to the shrinking of the perpendicular-space component of the diffraction vector by powers of $(-\tau)^{-n}$ while expanding the parallel-space component by τ^n according to the eigenvalues τ and $-\tau^{-1}$ of S acting in the two eigenspaces $\mathbf{V}^{\parallel}$ and $\mathbf{V}^{\perp}$:

$$\begin{aligned} \pi^{\parallel}(S\mathbf{H}) &= (h_2 + \tau(h_1 + h_2))a^* = (\tau h_1 + h_2(\tau + 1))a^* \\ &= \tau(h_1 + \tau h_2)a^*, \\ \pi^{\perp}(S\mathbf{H}) &= (-\tau h_2 + h_1 + h_2)a^* = (h_1 - h_2(\tau - 1))a^* \\ &= -(1/\tau)(-\tau h_1 + h_2)a^*, \end{aligned}$$

$$|F(\tau^n\mathbf{H}^{\parallel})| > |F(\tau^{n-1}\mathbf{H}^{\parallel})| > \ldots > |F(\tau\mathbf{H}^{\parallel})| > |F(\mathbf{H}^{\parallel})|.$$

Thus, for scaling n times we obtain

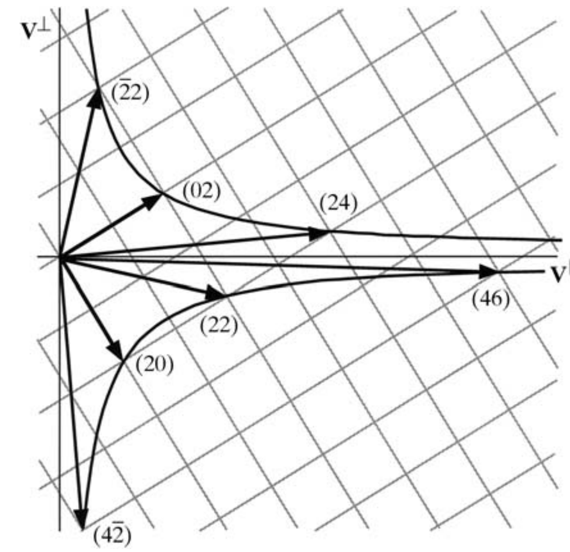

Fig. 4.6.3.11. Scaling operations of the Fibonacci sequence. The scaling operation S acts six times on the diffraction vector $\mathbf{H} = (4\bar{2})$ yielding the sequence $(4\bar{2}) \rightarrow (\bar{2}2) \rightarrow (20) \rightarrow (02) \rightarrow (22) \rightarrow (24) \rightarrow (46)$.

$$\begin{aligned} \pi^{\perp}(S^n\mathbf{H}) &= \big(-\tau(F_n h_1 + F_{n+1} h_2) + (F_{n+1} h_1 + F_{n+2} h_2)\big)a^* \\ &= \big(h_1(-\tau F_n + F_{n+1}) + h_2(-\tau F_{n+1} + F_{n+2})\big)a^* \end{aligned}$$

with

$$\lim_{n\to\infty}(-\tau F_n + F_{n+1}) = 0 \text{ and } \lim_{n\to\infty}(-\tau F_{n+1} + F_{n+2}) = 0,$$

yielding eventually

$$\lim_{n\to\infty}(\pi^{\perp}(S^n\mathbf{H})) = 0 \text{ and } \lim_{n\to\infty}(F(S^n\mathbf{H})) = F(\mathbf{0}).$$

The scaling of the diffraction vectors $\mathbf{H}$ by S^n corresponds to a hyperbolic rotation (Janner, 1992) with angle $n\varphi$, where $\sinh\varphi = 1/2$ (Fig. 4.6.3.11):

$$\begin{pmatrix} 0 & 1 \\ 1 & 1 \end{pmatrix}^{2n} = \begin{pmatrix} \cosh 2n\varphi & \sinh 2n\varphi \\ \sinh 2n\varphi & \cosh 2n\varphi \end{pmatrix},$$

$$\begin{pmatrix} 0 & 1 \\ 1 & 1 \end{pmatrix}^{2n+1} = \begin{pmatrix} \sinh[(2n+1)\varphi] & \cosh[(2n+1)\varphi] \\ \cosh[(2n+1)\varphi] & \sinh[(2n+1)\varphi] \end{pmatrix}.$$

4.6.3.3.2. Decagonal phases

A structure quasiperiodic in two dimensions, periodic in the third dimension and with decagonal diffraction symmetry is called a decagonal phase. Its holohedral Laue symmetry group is $K = 10/mmm$. All reciprocal-space vectors $\mathbf{H} \in M^*$ can be represented on a basis (V basis) $\mathbf{a}_i^* = a_i^*(\cos 2\pi i/5, \sin 2\pi i/5, 0)$, $i = 1, \ldots, 4$ and $\mathbf{a}_5^* = a_5^*(0, 0, 1)$ (Fig. 4.6.3.12) as $\mathbf{H} = \sum_{i=1}^{5} h_i \mathbf{a}_i^*$. The vector components refer to a Cartesian coordinate system in physical (parallel) space. Thus, from the number of independent reciprocal-basis vectors necessary to index the Bragg reflections with integer numbers, the dimension of the embedding space has to be at least five. This can also be shown in a different way (Hermann, 1949).

The set M^* of all vectors $\mathbf{H}$ remains invariant under the action of the symmetry operators of the point group $10/mmm$. The symmetry-adapted matrix representations for the point-group