International Tables for Crystallography (2010). Vol. B, Figure 4.6.3.14, p. 609.

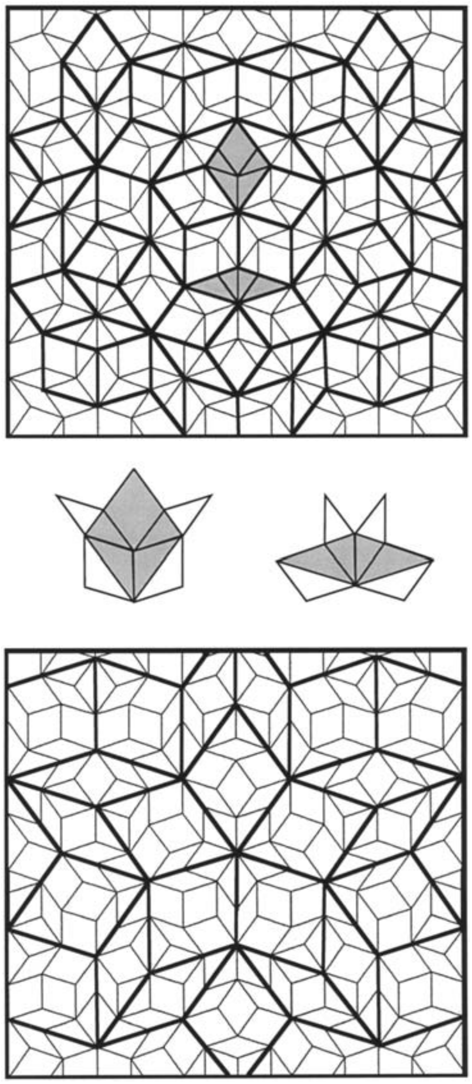

Fig. 4.6.3.13. A section of a Penrose tiling (thin lines) superposed by its τ-deflated tiling (above, thick lines) and by its τ^2-deflated tiling (below, thick lines). In the middle, the inflation rule of the Penrose tiling is illustrated.

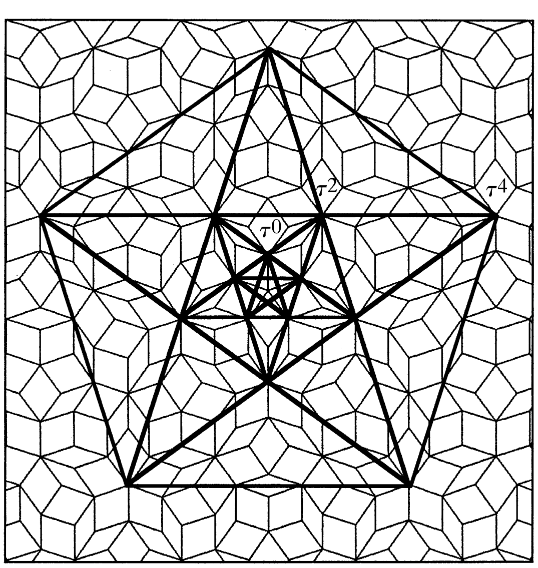

Fig. 4.6.3.14. Scaling symmetry of a pentagram superposed on the Penrose tiling. A vector pointing to a corner of a pentagon (star) is mapped by the rotoscaling operation (rotation around $\pi/5$ and dilatation by a factor τ^2) onto the next largest pentagon (star).

example, the canonical *Penrose tiling* (Penrose, 1974). It can be constructed from two unit tiles: a skinny (acute angle $\alpha_s = \pi/5$) and a fat (acute angle $\alpha_f = 2\pi/5$) rhomb with equal edge lengths a_r and areas $A_S = a_r^2 \sin(\pi/5)$, $A_F = a_r^2 \sin(2\pi/5)$ (Fig. 4.6.3.13). The areas and frequencies of these two unit tiles in the Penrose tiling are both in a ratio 1 to τ. By replacing each skinny and fat rhomb according to the inflation rule, a τ-inflated tiling is obtained. Inflation (deflation) means that the number of tiles is inflated (deflated), their edge lengths are decreased (increased) by a factor τ. By infinite repetition of this inflation operation, an infinite Penrose tiling is generated. Consequently, this substitution operation leaves the tiling invariant.

From Fig. 4.6.3.13 it can be seen that the sets of vertices of the deflated tilings are subsets of the set of vertices of the original tiling. The τ-deflated tiling is dual to the original tiling; a further deflation by a factor τ gives the original tiling again. However, the edge lengths of the tiles are increased by a factor τ^2, and the tiling is rotated around 36°. Only the fourth deflation of the original tiling yields the original tiling in its original orientation but with all lengths multiplied by a factor τ^4.

Contrary to the reciprocal-space scaling behaviour of $M^* = \{\mathbf{H}^\parallel = \sum_{i=1}^4 h_i \mathbf{a}_i^* | h_i \in \mathbb{Z}\}$, the set of vertices $M = \{\mathbf{r} = \sum_{i=1}^4 n_i \mathbf{a}_i | n_i \in \mathbb{Z}\}$ of the Penrose tiling is not invariant by scaling the length scale simply by a factor τ using the scaling matrix S:

$$S = \begin{pmatrix} 0 & 1 & 0 & \bar{1} \\ 0 & 1 & 1 & \bar{1} \\ \bar{1} & 1 & 1 & 0 \\ \bar{1} & 0 & 1 & 0 \end{pmatrix}_D \quad \text{acting on vectors } \mathbf{r} = \begin{pmatrix} n_1 \\ n_2 \\ n_3 \\ n_4 \end{pmatrix}_D .$$

The square of S, however, maps all vertices of the Penrose tiling upon other ones:

$$S^2 = \begin{pmatrix} 1 & 1 & 0 & \bar{1} \\ 0 & 2 & 1 & \bar{1} \\ \bar{1} & 1 & 2 & 0 \\ \bar{1} & 0 & 1 & 1 \end{pmatrix}_D , \quad \Gamma(\alpha)S^2 = \begin{pmatrix} 1 & 1 & \bar{1} & \bar{1} \\ 1 & 2 & 0 & \bar{2} \\ 0 & 2 & 1 & \bar{1} \\ \bar{1} & 1 & 1 & 0 \end{pmatrix}_D .$$

S^2 corresponds to a hyperbolic rotation with $\chi = \cosh^{-1}(3/2)$ in superspace (Janner, 1992). However, only operations of the type S^{4n}, $n = 0, 1, 2 \ldots$, scale the Penrose tiling in a way which is equivalent to the (4nth) substitutional operations discussed above. The rotoscaling operation $\Gamma(\alpha)S^2$, also a symmetry operation of the Penrose tiling, leaves a pentagram invariant as demonstrated in Fig. 4.6.3.14 (Janner, 1992). Block-diagonalization of the scaling matrix S decomposes it into two non-equivalent irreducible representations which give the scaling properties in the two orthogonal subspaces of the 4D embedding space, $\mathbf{V} = \mathbf{V}^\parallel \oplus \mathbf{V}^\perp$, the 2D parallel (physical) subspace $\mathbf{V}^\parallel$ and the perpendicular 2D subspace $\mathbf{V}^\perp$. Thus, using $WSW^{-1} = S_V = S_V^\parallel \oplus S_V^\perp$, we obtain

$$S_V = \left(\begin{array}{cc|cc} \tau & 0 & 0 & 0 \\ 0 & \tau & 0 & 0 \\ \hline 0 & 0 & -1/\tau & 0 \\ 0 & 0 & 0 & -1/\tau \end{array}\right)_V = \left(\begin{array}{c|c} S_V^\parallel & 0 \\ \hline 0 & S_V^\perp \end{array}\right)_V ,$$

where

references