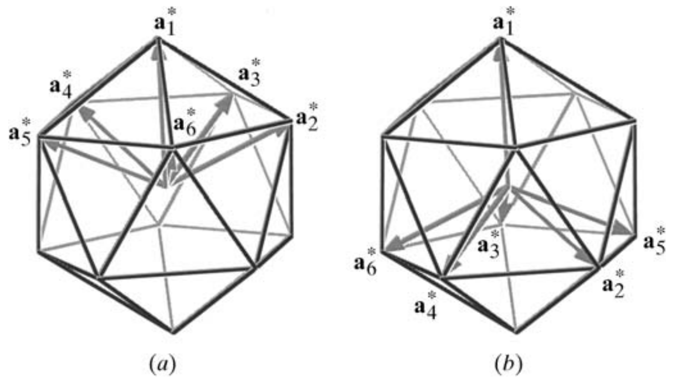

Fig. 4.6.3.28. Perspective (*a*) parallel- and (*b*) perpendicular-space views of the reciprocal basis of the 3D Penrose tiling. The six rationally independent vectors $\mathbf{a}_i^*$ point to the edges of an icosahedron.

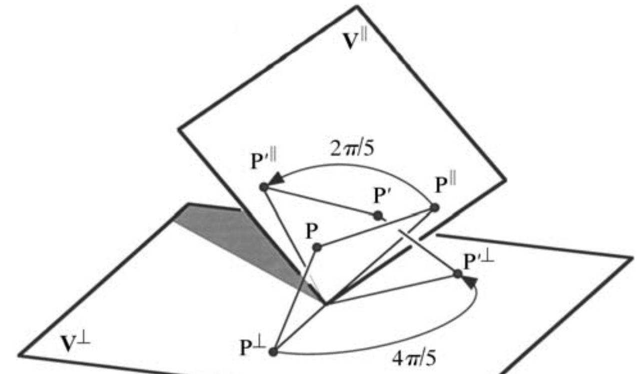

Fig. 4.6.3.29. Schematic representation of a rotation in 6D space. The point P is rotated to P′. The component rotations in parallel and perpendicular space are illustrated.

fold rotation α, a threefold rotation β and the inversion operation γ, can be written in the form

$$\Gamma(\alpha)=\begin{pmatrix}1&0&0&0&0&0\\0&0&0&0&0&1\\0&1&0&0&0&0\\0&0&1&0&0&0\\0&0&0&1&0&0\\0&0&0&0&1&0\end{pmatrix}_D,\ \Gamma(\beta)=\begin{pmatrix}0&1&0&0&0&0\\0&0&0&0&0&1\\0&0&0&\bar{1}&0&0\\0&0&0&0&\bar{1}&0\\0&0&1&0&0&0\\1&0&0&0&0&0\end{pmatrix}_D,$$

$$\Gamma(\gamma)=\begin{pmatrix}\bar{1}&0&0&0&0&0\\0&\bar{1}&0&0&0&0\\0&0&\bar{1}&0&0&0\\0&0&0&\bar{1}&0&0\\0&0&0&0&\bar{1}&0\\0&0&0&0&0&\bar{1}\end{pmatrix}_D.$$

Block-diagonalization of these reducible symmetry matrices decomposes them into non-equivalent irreducible representations. These can be assigned to the two orthogonal subspaces forming the 6D embedding space $\mathbf{V}=\mathbf{V}^{\parallel}\oplus\mathbf{V}^{\perp}$, the 3D parallel (physical) subspace $\mathbf{V}^{\parallel}$ and the perpendicular 3D subspace $\mathbf{V}^{\perp}$. Thus, using $W\Gamma W^{-1}=\Gamma^{\rm red}=\Gamma^{\parallel}\oplus\Gamma^{\perp}$, we obtain

$$\Gamma(\alpha)=\left(\begin{array}{ccc|ccc}\cos(2\pi/5)&-\sin(2\pi/5)&0&0&0&0\\\sin(2\pi/5)&\cos(2\pi/5)&0&0&0&0\\0&0&1&0&0&0\\\hline 0&0&0&\cos(4\pi/5)&-\sin(4\pi/5)&0\\0&0&0&\sin(4\pi/5)&\cos(4\pi/5)&0\\0&0&0&0&0&1\end{array}\right)_V$$
$$=\left(\begin{array}{c|c}\Gamma^{\parallel}&0\\\hline 0&\Gamma^{\perp}\end{array}\right)_V,$$

where

$$W=a^*\left(\begin{array}{cccccc}0&sc4&sc6&sc8&s&sc2\\0&ss4&ss6&ss8&0&ss2\\1&c&c&c&c&c\\\hline 0&-sc8&-sc2&-sc6&-s&-sc4\\0&-ss8&-ss2&-ss6&0&-ss4\\1&-c&-c&-c&-c&-c\end{array}\right)_V,$$

$c=\cos\theta,\ s=\sin\theta,\ scn=\sin\theta\cos(n\pi/5),\ ssn=\sin\theta\sin(n\pi/5)$. The column vectors of the matrix W give the parallel- (above the partition line) and perpendicular-space components (below the partition line) of a reciprocal basis in $\mathbf{V}$. Thus, W can be rewritten using the physical-space reciprocal basis defined above and an arbitrary constant c,

$$W=\begin{pmatrix}\mathbf{a}_1^*&\mathbf{a}_2^*&\mathbf{a}_3^*&\mathbf{a}_4^*&\mathbf{a}_5^*&\mathbf{a}_6^*\\c\mathbf{a}_1^*&-c\mathbf{a}_4^*&-c\mathbf{a}_6^*&-c\mathbf{a}_3^*&-c\mathbf{a}_5^*&-c\mathbf{a}_2^*\end{pmatrix}$$
$$=\begin{pmatrix}\mathbf{d}_1^*&\mathbf{d}_2^*&\mathbf{d}_3^*&\mathbf{d}_4^*&\mathbf{d}_5^*&\mathbf{d}_6^*\end{pmatrix},$$

yielding the reciprocal basis $\mathbf{d}_i^*$, $i=1,\ldots,6$, in the 6D embedding space (D space)

$$\mathbf{d}_1^*=a^*\begin{pmatrix}0\\0\\1\\0\\0\\c\end{pmatrix}\ \text{and}\ \mathbf{d}_i^*=a^*\begin{pmatrix}\sin\theta\cos(2\pi i/5)\\\sin\theta\sin(2\pi i/5)\\\cos\theta\\-c\sin\theta\cos(4\pi i/5)\\-c\sin\theta\sin(4\pi i/5)\\-c\cos\theta\end{pmatrix},\ i=2,\ldots,6.$$

The 6×6 symmetry matrices can each be decomposed into two 3×3 matrices. The first one, $\Gamma^{\parallel}$, acts on the parallel-space component, the second one, $\Gamma^{\perp}$, on the perpendicular-space component. In the case of $\Gamma(\alpha)$, the coupling factor between a rotation in parallel and perpendicular space is 2. Thus a $2\pi/5$ rotation in physical space is related to a $4\pi/5$ rotation in perpendicular space (Figs. 4.6.3.28 and 4.6.3.29).

With the condition $\mathbf{d}_i\cdot\mathbf{d}_j^*=\delta_{ij}$, the basis in direct 6D space is obtained:

$$\mathbf{d}_1=\frac{1}{2a^*}\begin{pmatrix}0\\0\\1\\0\\0\\1/c\end{pmatrix}\ \text{and}\ \mathbf{d}_i=\frac{1}{2a^*}\begin{pmatrix}\sin\theta\cos(2\pi i/5)\\\sin\theta\sin(2\pi i/5)\\\cos\theta\\-(1/c)\sin\theta\cos(4\pi i/5)\\-(1/c)\sin\theta\sin(4\pi i/5)\\-(1/c)\cos\theta\end{pmatrix},$$
$$i=2,\ldots,6.$$

The metric tensors G, G^* are of the type