Table 9.1.1.1. *Types of circle packings in the plane*

| Type | $k$ | Symmetry | | | Parameters | Distance $d$ | Density |
|---|---|---|---|---|---|---|---|
| $3^6$ | 6 | $p6mm$ | $1(a)$ | $0, 0$ | | $a$ | 0.9069 |
| $3^2434$ | 5 | $p4gm$ | $4(c)$ | $x, x+\frac{1}{2}$ | $x=\frac{1}{4}\sqrt{3}-\frac{1}{4}$ | $\frac{1}{2}(\sqrt{6}-\sqrt{2})a$ | 0.8418 |
| $3^34^2$ | 5 | $c2mm$ | $4(d)$ | $x, 0$ | $x=1-\frac{1}{2}\sqrt{3}$; $b/a=2-\sqrt{3}$ | $b$ | 0.8418 |
| $3^46$ | 5 | $p6$ | $6(d)$ | $x, y$ | $x=\frac{3}{7}$; $y=\frac{1}{7}$ | $\frac{1}{7}\sqrt{7}a$ | 0.7773 |
| $4^4$ | 4 | $p4mm$ | $1(a)$ | $0, 0$ | | $a$ | 0.7854 |
| 3464 | 4 | $p6mm$ | $6(e)$ | $x, \bar{x}$ | $x=\frac{1}{2}-\frac{1}{6}\sqrt{3}$ | $\frac{1}{2}(\sqrt{3}-1)a$ | 0.7290 |
| 3636 | 4 | $p6mm$ | $3(c)$ | $\frac{1}{2}, 0$ | | $\frac{1}{2}a$ | 0.6802 |
| $6^3$ | 3 | $p6mm$ | $2(b)$ | $\frac{1}{3}, \frac{2}{3}$ | | $\frac{1}{3}\sqrt{3}a$ | 0.6046 |
| $48^2$ | 3 | $p4mm$ | $4(d)$ | $x, 0$ | $x=1-\frac{1}{2}\sqrt{2}$ | $(\sqrt{2}-1)a$ | 0.5390 |
| 46.12 | 3 | $p6mm$ | $12(f)$ | $x, y$ | $x=\frac{1}{6}\sqrt{3}+\frac{1}{6}$; $y=\frac{1}{6}\sqrt{3}-\frac{1}{6}$ | $(\frac{1}{2}-\frac{1}{6}\sqrt{3})a$ | 0.4860 |
| $3.12^2$ | 3 | $p6mm$ | $6(c)$ | $x, \bar{x}$ | $x=1-\frac{1}{3}\sqrt{3}$ | $(2-\sqrt{3})a$ | 0.3907 |

Table 9.1.1.2 shows examples for sphere packings with high contact numbers and high densities in the upper part and with small contact numbers and low densities in the lower part. Column 1 gives reference numbers to designate the types in the following. Column 2 displays the contact numbers $k$. The highest possible symmetry for each type is described in column 3. Coordinates and metrical parameters referring to the most regular sphere packings of each type are listed in column 4; the respective shortest distances $d$ between sphere centres are given in column 5. For a sphere packing that can be subdivided into plane nets of spheres with mutual contact, the direction and the type of these nets are shown in column 6. Column 7 contains stacking information: the contact numbers to the nets above and below, and the number of layers per translation period in the direction perpendicular to the layers. The last column displays the density with respect to the parameters of column 4. For all cases, this value gives the minimal density for that type of sphere packing.

The densest homogeneous sphere packings known so far may be derived from the densest packings of circles ($3^6$ in Table 9.1.1.1). Such sphere packings can always be subdivided into parallel plane layers of spheres with six contacts per sphere within each layer and with three contacts to each of the neighbouring layers above and below (*cf.* Fig. 9.1.1.1). Consequently, the contact number $k$ becomes 12. As there exist two stacking possibilities for each layer with respect to the previous layer, infinitely many stacking sequences can be derived in principle, but only two refer to homogeneous sphere packings. If for each layer the two neighbouring layers are stacked directly upon each other, a sphere packing of a two-layer type with hexagonal symmetry (type 1) results. It is called *hexagonal closest packing* (abbreviated h.c.p.). If for all layers the neighbouring layers are never stacked directly upon each other, a sphere packing of a three-layer type with cubic symmetry (type 2) is formed. It is designated *cubic closest packing* (c.c.p.). In spite of these terms, for a long time it was only known that the cubic closest packings are the densest ones that correspond to lattices (Minkowski, 1904). Only recently, Hsiang (1993) published a proof that there does not exist any packing of spheres of equal size with a higher density, but the completeness of this proof is still doubted (*cf. e.g.* Hales, 1994).

Independently of the stacking sequences, closest packings of spheres contain ideal octahedral and ideal tetrahedral voids. The number of octahedra per unit cell equals the respective number of spheres, whereas the number of tetrahedral voids is twice as large. The distances between the centres and the vertices of these voids are $\sqrt{2}d/2$ and $\sqrt{6}d/4$, respectively. Within a cubic closest packing, faces are shared only between octahedral and tetrahedral voids. Each edge is common to two octahedra and two tetrahedra. In contrast, piles of face-sharing octahedra are formed within a hexagonal closest packing, whereas the tetrahedra are arranged as pairs with one face in common. The other faces are shared between octahedra and tetrahedra. Again, each edge belongs to two octahedra and two tetrahedra.

Densest layers of spheres may also be stacked such that each sphere is in contact with two spheres of the previous layers (*cf.* Fig. 9.1.1.2). Such a stacking results in contact number 10. Again, infinitely many periodic stacking sequences are possible, but only four give rise to homogeneous sphere packings [types 9, 10, 11: *cf.* Hellner (1986); type 12: *cf.* O'Keeffe (1988)]. In the most symmetrical forms of these four cases, each sphere is located exactly above or below the middle of two neighbouring spheres of the adjacent layers. This kind of stacking gives rise to distorted tetrahedral voids only. The number of tetrahedra per unit cell is six times the number of spheres. Two kinds of differently distorted tetrahedra exist in the ratio 1:2. The two-layer type 9 corresponds to a tetragonal body-centred lattice with specialized axial ratio.

Furthermore, densest layers of spheres may be stacked in a mixed sequence with three contacts per sphere to one neighbouring layer and two contacts to the other layer. This kind of stacking results in five types of homogeneous sphere packings (3 to 7) with contact number 11.

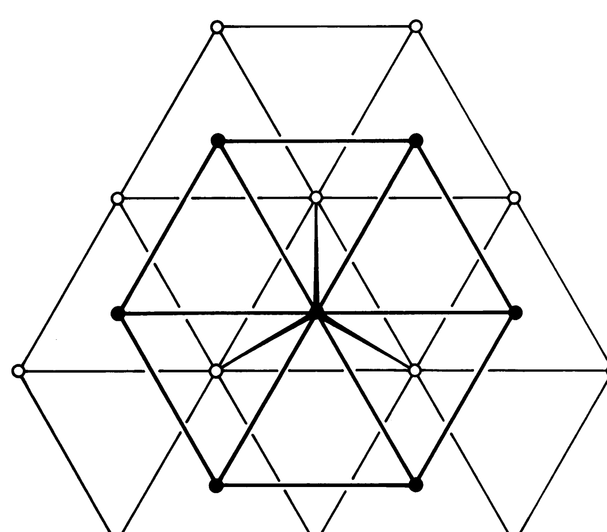

Fig. 9.1.1.1. Two triangular nets representing two densest packed layers of spheres. The layers are stacked in such a way that each sphere is in contact with three spheres of the other layer.



 references