International Tables for Crystallography (2006). Vol. D, Section 1.1.3.5.2, pp. 8–9.

1.1.3.3.2. *Multiplication by a scalar*

This is a particular case of the tensor product.

1.1.3.3.3. *Contracted product, contraction*

Here we are concerned with an operation that only exists in the case of tensors and that is very important because of its applications in physics. In practice, it is almost always the case that tensors enter into physics through the intermediary of a contracted product.

(i) *Contraction*. Let us consider a tensor of rank 2 that is once covariant and once contravariant. Let us write its transformation in a change of coordinate system:

$$t'^{j}_{i} = A^{j}_{p} B^{q}_{i} t^{p}_{q}.$$

Now consider the quantity t'^{i}_{i} derived by applying the Einstein convention ($t'^{i}_{i} = t'^{1}_{1} + t'^{2}_{2} + t'^{3}_{3}$). It follows that

$$t'^{i}_{i} = A^{i}_{p} B^{q}_{i} t^{p}_{q} = \delta^{q}_{p} t^{q}_{q}$$
$$t'^{i}_{i} = t^{p}_{p}.$$

This is an invariant quantity and so is a *scalar*. This operation can be carried out on any tensor of rank higher than or equal to two, provided that it is expressed in a form such that its components are (at least) once covariant and once contravariant.

The *contraction* consists therefore of equalizing a covariant index and a contravariant index, and then in summing over this index. Let us take, for example, the tensor t'^{jk}_{i}. Its contracted form is t'^{ik}_{i}, which, with a change of basis, becomes

$$t'^{ik}_{i} = A^{k}_{p} t^{qp}_{q}.$$

The components t^{ik}_{i} are those of a vector, resulting from the contraction of the tensor t^{jk}_{i}. The rank of the tensor has changed from 3 to 1. In a general manner, the contraction reduces the rank of the tensor from n to $n-2$.

Example. Let us take again the operator of symmetry O. The trace of the associated matrix is equal to

$$O^{1}_{1} + O^{2}_{2} + O^{3}_{3} = O^{i}_{i}.$$

It is the resultant of the contraction of the tensor O. It is a tensor of rank 0, which is a scalar and is invariant under a change of basis.

(ii) *Contracted product*. Consider the product of two tensors of which one is contravariant at least once and the other covariant at least once:

$$p^{jk}_{i} = t^{j}_{i} z^{k}.$$

If we contract the indices i and k, it follows that

$$p^{ji}_{i} = t^{j}_{i} z^{i}.$$

The contracted product is then a tensor of rank 1 and not 3. It is an operation that is very frequent in practice.

(iii) *Scalar product*. Next consider the tensor product of two vectors:

$$t^{j}_{i} = x_{i} y^{j}.$$

After contraction, we get the scalar product:

$$t^{i}_{i} = x_{i} y^{i}.$$

1.1.3.4. *Tensor nature of physical quantities*

Let us first consider the dielectric constant. In the introduction, we remarked that for an isotropic medium

$$\mathbf{D} = \varepsilon \mathbf{E}.$$

If the medium is anisotropic, we have, for one of the components,

$$D^{1} = \varepsilon^{1}_{1} E^{1} + \varepsilon^{1}_{2} E^{2} + \varepsilon^{1}_{3} E^{3}.$$

This relation and the equivalent ones for the other components can also be written

$$D^{i} = \varepsilon^{i}_{j} E^{j} \qquad (1.1.3.3)$$

using the Einstein convention.

The scalar product of $\mathbf{D}$ by an arbitrary vector $\mathbf{x}$ is

$$D^{i} x_{i} = \varepsilon^{i}_{j} E^{j} x_{i}.$$

The right-hand member of this relation is a bilinear form that is invariant under a change of basis. The set of nine quantities ε^{i}_{j} constitutes therefore the set of components of a tensor of rank 2. Expression (1.1.3.3) is the contracted product of ε^{i}_{j} by E^{j}.

A similar demonstration may be used to show the tensor nature of the various physical properties described in Section 1.1.1, whatever the rank of the tensor. Let us for instance consider the piezoelectric effect (see Section 1.1.4.4.3). The components of the electric polarization, P^{i}, which appear in a medium submitted to a stress represented by the second-rank tensor T_{jk} are

$$P^{i} = d^{ijk} T_{jk},$$

where the tensor nature of T_{jk} will be shown in Section 1.3.2. If we take the contracted product of both sides of this equation by any vector of covariant components x_{i}, we obtain a linear form on the left-hand side, and a trilinear form on the right-hand side, which shows that the coefficients d^{ijk} are the components of a third-rank tensor. Let us now consider the piezo-optic (or photoelastic) effect (see Sections 1.1.4.10.5 and 1.6.7). The components of the variation $\Delta\eta^{ij}$ of the dielectric impermeability due to an applied stress are

$$\Delta\eta^{ij} = \pi^{ijkl} T_{jl}.$$

In a similar fashion, consider the contracted product of both sides of this relation by two vectors of covariant components x_{i} and y_{j}, respectively. We obtain a bilinear form on the left-hand side, and a quadrilinear form on the right-hand side, showing that the coefficients π^{ijkl} are the components of a fourth-rank tensor.

1.1.3.5. *Representation surface of a tensor*

1.1.3.5.1. *Definition*

Let us consider a tensor $t_{ijkl\ldots}$ represented in an orthonormal frame where variance is not important. The value of component $t'_{1111\ldots}$ in an arbitrary direction is given by

$$t'_{1111\ldots} = t_{ijkl\ldots} B^{i}_{1} B^{j}_{1} B^{k}_{1} B^{l}_{1} \ldots,$$

where the B^{i}_{1}, $B^{j}_{1}, \ldots$ are the direction cosines of that direction with respect to the axes of the orthonormal frame.

The *representation surface* of the tensor is the polar plot of $t'_{1111\ldots}$.

1.1.3.5.2. *Representation surfaces of second-rank tensors*

The representation surfaces of second-rank tensors are quadrics. The directions of their principal axes are obtained as follows. Let t_{ij} be a second-rank tensor and let $\mathbf{OM} = \mathbf{r}$ be a vector with coordinates x_{i}. The doubly contracted product, $t_{ij} x^{i} x^{j}$, is a scalar. The locus of points M such that

$$t_{ij}x^i x^j = 1$$

is a quadric. Its principal axes are along the directions of the eigenvectors of the matrix with elements t_{ij}. They are solutions of the set of equations

$$t_{ij}x^i = \lambda x^j,$$

where the associated quantities λ are the eigenvalues.

Let us take as axes the principal axes. The equation of the quadric reduces to

$$t_{11}(x^1)^2 + t_{22}(x^2)^2 + t_{33}(x^3)^2 = 1.$$

If the eigenvalues are all of the same sign, the quadric is an ellipsoid; if two are positive and one is negative, the quadric is a hyperboloid with one sheet; if one is positive and two are negative, the quadric is a hyperboloid with two sheets (see Section 1.3.1).

Associated quadrics are very useful for the geometric representation of physical properties characterized by a tensor of rank 2, as shown by the following examples:

(i) *Index of refraction* of a medium. It is related to the dielectric constant by $n = \varepsilon^{1/2}$ and, like it, it is a tensor of rank 2. Its associated quadric is an ellipsoid, the optical indicatrix, which represents its variations with the direction in space (see Section 1.6.3.2).

(ii) *Thermal expansion*. If one cuts a sphere in a medium whose thermal expansion is anisotropic, and if one changes the temperature, the sphere becomes an ellipsoid. Thermal expansion is therefore represented by a tensor of rank 2 (see Chapter 1.4).

(iii) *Thermal conductivity*. Let us place a drop of wax on a plate of gypsum, and then apply a hot point at the centre. There appears a halo where the wax has melted: it is elliptical, indicating anisotropic conduction. Thermal conductivity is represented by a tensor of rank 2 and the elliptical halo of molten wax corresponds to the intersection of the associated ellipsoid with the plane of the plate of gypsum.

1.1.3.5.3. *Representation surfaces of higher-rank tensors*

Examples of representation surfaces of higher-rank tensors are given in Sections 1.3.3.4.4 and 1.9.4.2.

1.1.3.6. *Change of variance of the components of a tensor*

1.1.3.6.1. *Tensor nature of the metric tensor*

Equation (1.1.2.17) describing the behaviour of the quantities $g_{ij} = \mathbf{e}_i \cdot \mathbf{e}_j$ under a change of basis shows that they are the components of a tensor of rank 2, the *metric tensor*. In the same way, equation (1.1.2.19) shows that the g^{ij}'s transform under a change of basis like the product of two contravariant coordinates. The coefficients g^{ij} and g_{ij} are the components of a *unique tensor*, in one case doubly contravariant, in the other case doubly covariant. In a general way, the Euclidean tensors (constructed in a space where one has defined the scalar product) are geometrical entities that can have covariant, contravariant or mixed components.

1.1.3.6.2. *How to change the variance of the components of a tensor*

Let us take a tensor product

$$t^{ij} = x^i y^j.$$

We know that

$$x^i = g^{ik}x_k \quad \text{and} \quad y^j = g^{jl}y_l.$$

It follows that

$$t^{ij} = g^{ik}g^{jl}x_k y_l.$$

$x_k y_l$ is a tensor product of two vectors expressed in the dual space:

$$x_k y_l = t_{kl}.$$

One can thus pass from the doubly covariant form to the doubly contravariant form of the tensor by means of the relation

$$t^{ij} = g^{ik}g^{jl}t_{kl}.$$

This result is general: to change the variance of a tensor (in practice, to raise or lower an index), it is necessary to make the contracted product of this tensor using g^{ij} or g_{ij}, according to the case. For instance,

$$t_k^l = g^{jl}t_{lk}; \quad t_k^{ij} = g_{kl}t^{ijl}.$$

Remark

$$g_j^i = g^{ik}g_{kj} = \delta_j^i.$$

This is a property of the metric tensor.

1.1.3.6.3. *Examples of the use in physics of different representations of the same quantity*

Let us consider, for example, the force, $\mathbf{F}$, which is a tensor quantity (tensor of rank 1). One can define it:

(i) by the fundamental law of dynamics:

$$\mathbf{F} = m\boldsymbol{\Gamma}, \quad \text{with } F^i = m\, \mathrm{d}^2x^i/\mathrm{d}t^2,$$

where m is the mass and $\boldsymbol{\Gamma}$ is the acceleration. The force appears here in a *contravariant* form.

(ii) as the derivative of the energy, W:

$$F_i = \partial W/\partial x^i = \partial_i W.$$

The force appears here in *covariant* form. In effect, we shall see in Section 1.1.3.8.1 that to form a derivative with respect to a variable contravariant augments the covariance by unity. The general expression of the law of dynamics is therefore written with the energy as follows:

$$m\, \mathrm{d}^2x^i/\mathrm{d}t^2 = g^{ij}\partial_j W.$$

1.1.3.7. *Outer product*

1.1.3.7.1. *Definition*

The tensor defined by

$$\mathbf{x} \bigwedge \mathbf{y} = \mathbf{x} \otimes \mathbf{y} - \mathbf{y} \otimes \mathbf{x}$$

is called the *outer product* of vectors $\mathbf{x}$ and $\mathbf{y}$. (*Note*: The symbol is different from the symbol $\wedge$ for the vector product.) The analytical expression of this tensor of rank 2 is

$$\left.\begin{aligned} \mathbf{x} &= x^i\mathbf{e}_i \\ \mathbf{y} &= y^j\mathbf{e}_j \end{aligned}\right\} \implies \mathbf{x} \bigwedge \mathbf{y} = (x^i y^j - y^i x^j)\,\mathbf{e}_i \otimes \mathbf{e}_j.$$

The components $p^{ij} = x^i y^j - y^i x^j$ of this tensor satisfy the properties

$$p^{ij} = -p^{ji}; \quad p^{ii} = 0.$$

It is an *antisymmetric* tensor of rank 2.