$$t_{ij}x^i x^j = 1$$

is a quadric. Its principal axes are along the directions of the eigenvectors of the matrix with elements t_{ij}. They are solutions of the set of equations

$$t_{ij}x^i = \lambda x^j,$$

where the associated quantities λ are the eigenvalues.

Let us take as axes the principal axes. The equation of the quadric reduces to

$$t_{11}(x^1)^2 + t_{22}(x^2)^2 + t_{33}(x^3)^2 = 1.$$

If the eigenvalues are all of the same sign, the quadric is an ellipsoid; if two are positive and one is negative, the quadric is a hyperboloid with one sheet; if one is positive and two are negative, the quadric is a hyperboloid with two sheets (see Section 1.3.1).

Associated quadrics are very useful for the geometric representation of physical properties characterized by a tensor of rank 2, as shown by the following examples:

(i) *Index of refraction* of a medium. It is related to the dielectric constant by $n = \varepsilon^{1/2}$ and, like it, it is a tensor of rank 2. Its associated quadric is an ellipsoid, the optical indicatrix, which represents its variations with the direction in space (see Section 1.6.3.2).

(ii) *Thermal expansion*. If one cuts a sphere in a medium whose thermal expansion is anisotropic, and if one changes the temperature, the sphere becomes an ellipsoid. Thermal expansion is therefore represented by a tensor of rank 2 (see Chapter 1.4).

(iii) *Thermal conductivity*. Let us place a drop of wax on a plate of gypsum, and then apply a hot point at the centre. There appears a halo where the wax has melted: it is elliptical, indicating anisotropic conduction. Thermal conductivity is represented by a tensor of rank 2 and the elliptical halo of molten wax corresponds to the intersection of the associated ellipsoid with the plane of the plate of gypsum.

1.1.3.5.3. *Representation surfaces of higher-rank tensors*

Examples of representation surfaces of higher-rank tensors are given in Sections 1.3.3.4.4 and 1.9.4.2.

1.1.3.6. *Change of variance of the components of a tensor*

1.1.3.6.1. *Tensor nature of the metric tensor*

Equation (1.1.2.17) describing the behaviour of the quantities $g_{ij} = \mathbf{e}_i \cdot \mathbf{e}_j$ under a change of basis shows that they are the components of a tensor of rank 2, the *metric tensor*. In the same way, equation (1.1.2.19) shows that the g^{ij}'s transform under a change of basis like the product of two contravariant coordinates. The coefficients g^{ij} and g_{ij} are the components of a *unique tensor*, in one case doubly contravariant, in the other case doubly covariant. In a general way, the Euclidean tensors (constructed in a space where one has defined the scalar product) are geometrical entities that can have covariant, contravariant or mixed components.

1.1.3.6.2. *How to change the variance of the components of a tensor*

Let us take a tensor product

$$t^{ij} = x^i y^j.$$

We know that

$$x^i = g^{ik}x_k \quad \text{and} \quad y^j = g^{jl}y_l.$$

It follows that

$$t^{ij} = g^{ik}g^{jl}x_k y_l.$$

$x_k y_l$ is a tensor product of two vectors expressed in the dual space:

$$x_k y_l = t_{kl}.$$

One can thus pass from the doubly covariant form to the doubly contravariant form of the tensor by means of the relation

$$t^{ij} = g^{ik}g^{jl}t_{kl}.$$

This result is general: to change the variance of a tensor (in practice, to raise or lower an index), it is necessary to make the contracted product of this tensor using g^{ij} or g_{ij}, according to the case. For instance,

$$t_k^l = g^{jl}t_{lk}; \quad t_k^{ij} = g_{kl}t^{ijl}.$$

Remark

$$g_j^i = g^{ik}g_{kj} = \delta_j^i.$$

This is a property of the metric tensor.

1.1.3.6.3. *Examples of the use in physics of different representations of the same quantity*

Let us consider, for example, the force, **F**, which is a tensor quantity (tensor of rank 1). One can define it:

(i) by the fundamental law of dynamics:

$$\mathbf{F} = m\boldsymbol{\Gamma}, \quad \text{with } F^i = m \,\mathrm{d}^2x^i/\mathrm{d}t^2,$$

where m is the mass and $\boldsymbol{\Gamma}$ is the acceleration. The force appears here in a *contravariant* form.

(ii) as the derivative of the energy, W:

$$F_i = \partial W/\partial x^i = \partial_i W.$$

The force appears here in *covariant* form. In effect, we shall see in Section 1.1.3.8.1 that to form a derivative with respect to a variable contravariant augments the covariance by unity. The general expression of the law of dynamics is therefore written with the energy as follows:

$$m \,\mathrm{d}^2x^i/\mathrm{d}t^2 = g^{ij}\partial_j W.$$

1.1.3.7. *Outer product*

1.1.3.7.1. *Definition*

The tensor defined by

$$\mathbf{x} \bigwedge \mathbf{y} = \mathbf{x} \otimes \mathbf{y} - \mathbf{y} \otimes \mathbf{x}$$

is called the *outer product* of vectors **x** and **y**. (*Note*: The symbol is different from the symbol $\wedge$ for the vector product.) The analytical expression of this tensor of rank 2 is

$$\left.\begin{array}{ll}\mathbf{x} & = x^i\mathbf{e}_i \\ \mathbf{y} & = y^j\mathbf{e}_j\end{array}\right\} \Longrightarrow \mathbf{x} \bigwedge \mathbf{y} = (x^i y^j - y^i x^j)\,\mathbf{e}_i \otimes \mathbf{e}_j.$$

The components $p^{ij} = x^i y^j - y^i x^j$ of this tensor satisfy the properties

$$p^{ij} = -p^{ji}; \quad p^{ii} = 0.$$

It is an *antisymmetric* tensor of rank 2.