International Tables for Crystallography (2006). Vol. D, Section 1.1.4.10.4.5, pp. 25–26.

$$\begin{pmatrix} \mathbf{1} \\ \hline \mathbf{2} \\ \hline \mathbf{3} \end{pmatrix}.$$

Both sides of relation (1.1.4.8) remain unchanged if the indices i and j are interchanged, on account of the symmetry of the stress tensor. This shows that

$$e_{ijk} = e_{jik}.$$

Submatrices **2** and **3** are equal. One introduces here a two-index notation through the relation $e_{\alpha k} = e_{ijk}$, and the $e_{\alpha k}$ matrix can be written

$$\left(\frac{\mathbf{1}}{\mathbf{2}+\mathbf{3}}\right).$$

The relation between the full and the reduced matrix is therefore different for the d_{ijk} and the e_{kij} tensors. This is due to the particular property of the strain Voigt matrix (1.1.4.6), and as a consequence the relations between nonzero components of the reduced matrices are different for certain point groups (3, 32, 3m, $\bar{6}$, $\bar{6}2m$).

1.1.4.10.4. *Independent components of the matrix associated with a third-rank polar tensor according to the following point groups*

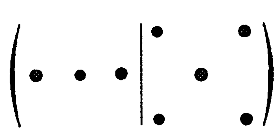
1.1.4.10.4.1. *Triclinic system*

(i) Group 1: all the components are independent. There are 18 components.

(ii) Group $\bar{1}$: all the components are equal to zero.

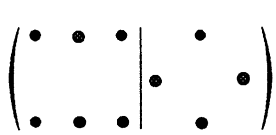
1.1.4.10.4.2. *Monoclinic system*

(i) Group 2: twofold axis parallel to Ox_2:



There are 8 independent components.

(ii) Group m:

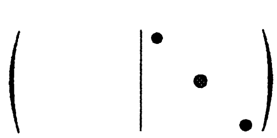


There are 10 independent components.

(iii) Group $2/m$: all the components are equal to zero.

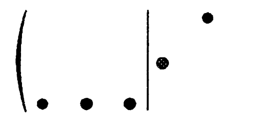
1.1.4.10.4.3. *Orthorhombic system*

(i) Group 222:



There are 3 independent components.

(ii) Group $mm2$:

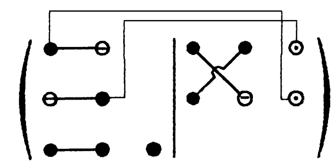


There are 5 independent components.

(iii) Group mmm: all the components are equal to zero.

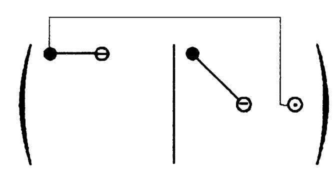
1.1.4.10.4.4. *Trigonal system*

(i) Group 3:



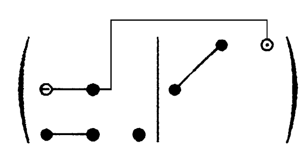
where the symbol ⊖ means that the corresponding component is equal to the opposite of that to which it is linked, ⊙ means that the component is equal to twice minus the value of the component to which it is linked for d_{ijk} and to minus the value of the component to which it is linked for e_{ijk}. There are 6 independent components.

(ii) Group 32, twofold axis parallel to Ox_1:



with the same conventions. There are 4 independent components.

(iii) Group 3m, mirror perpendicular to Ox_1:

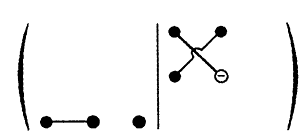


with the same conventions. There are 4 independent components.

(iv) Groups $\bar{3}$ and $\bar{3}m$: all the components are equal to zero.

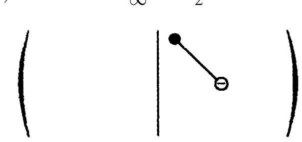
1.1.4.10.4.5. *Tetragonal, hexagonal and cylindrical systems*

(i) Groups 4, 6 and A_∞:



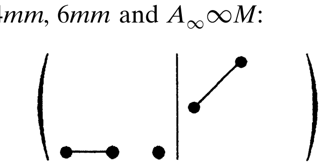
There are 4 independent components.

(ii) Groups 422, 622 and $A_\infty \infty A_2$:



There is 1 independent component.

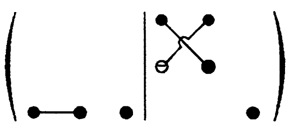
(iii) Groups 4mm, 6mm and $A_\infty \infty M$:



There are 3 independent components.

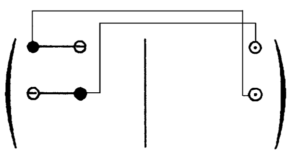
(iv) Groups $4/m$, $6/m$ and $(A_\infty/M)C$: all the components are equal to zero.

(v) Group $\bar{4}$:



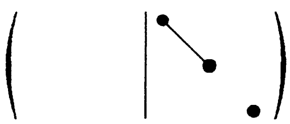
There are 4 independent components.

(vi) Group $\bar{6} = 3/m$:



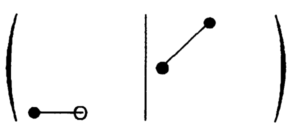
with the same conventions as for group 3. There are 2 independent components.

(vii) Group $\bar{4}2m$ – twofold axis parallel to Ox_1:



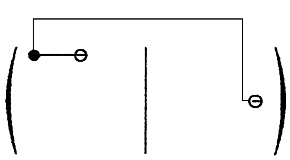
There are 2 independent components.

(viii) Group $\bar{4}2m$ – mirror perpendicular to Ox_1 (twofold axis at 45°):



The number of independent components is of course the same.

(ix) Group $\bar{6}2/m$:

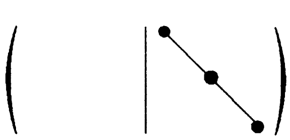


with the same conventions as for group 3. There is 1 independent component.

(x) Groups $4/mm$, $6/mm$ and $(A_\infty/M)\infty(A_2/M)C$: all the components are equal to zero.

1.1.4.10.4.6. *Cubic and spherical systems*

(i) Groups 23 and $\bar{4}3m$:



There is 1 independent component.

(ii) Groups 432 and ∞A_∞: it was seen in Section 1.1.4.8.6 that we have in this case

$$d_{123} = -d_{132}.$$

It follows that $d_{14} = 0$, all the components are equal to zero.

(iii) Groups $m3$, $m\bar{3}m$ and $\infty(A_\infty/M)C$: all the components are equal to zero.

1.1.4.10.5. *Reduction of the number of independent components of fourth-rank polar tensors due to the symmetry of the strain and stress tensors*

Let us consider five examples of fourth-rank tensors:

(i) *Elastic compliances*, s_{ijkl}, relating the resulting strain tensor S_{ij} to an applied stress T_{ij} (see Section 1.3.3.2):

$$S_{ij} = s_{ijkl}T_{kl}, \qquad (1.1.4.9)$$

where the compliances s_{ijkl} are the components of a tensor of rank 4.

(ii) *Elastic stiffnesses*, c_{ijkl} (see Section 1.3.3.2):

$$T_{ij} = c_{ijkl}S_{kl}.$$

(iii) *Piezo-optic coefficients*, π_{ijkl}, relating the variation $\Delta\eta_{ij}$ of the dielectric impermeability to an applied stress T_{kl} (*photoelastic effect* – see Section 1.6.7):

$$\Delta\eta_{ij} = \pi_{ijkl}T_{kl}.$$

(iv) *Elasto-optic coefficients*, p_{ijkl}, relating the variation $\Delta\eta_{ij}$ of the dielectric impermeability to the strain S_{kl}:

$$\Delta\eta_{ij} = p_{ijkl}S_{kl}.$$

(v) *Electrostriction coefficients*, Q_{ijkl}, which appear in equation (1.1.4.4):

$$S_{ij} = Q_{ijkl}E_kE_l, \qquad (1.1.4.10)$$

where only the second-order terms are considered.

In each of the equations from (1.1.4.9) to (1.1.4.10), the contracted product of a fourth-rank tensor by a symmetric second-rank tensor is equal to a symmetric second-rank tensor. As in the case of the third-rank tensors, this results in a reduction of the number of independent components, but because of the properties of the strain Voigt matrix, and because two of the tensors are endowed with intrinsic symmetry (the elastic tensors), the reduction is different for each of the five tensors. The above relations can be written in matrix form:

$$\begin{pmatrix} \mathbf{1} \\ \mathbf{2} \\ \mathbf{3} \end{pmatrix} = \begin{pmatrix} \mathbf{1} & \mathbf{2} & \mathbf{3} \\ \mathbf{4} & \mathbf{5} & \mathbf{6} \\ \mathbf{7} & \mathbf{8} & \mathbf{9} \end{pmatrix} \times \begin{pmatrix} \mathbf{1} \\ \mathbf{2} \\ \mathbf{3} \end{pmatrix}$$

where the second-rank tensors are represented by 1×9 column matrices, which can each be subdivided into three 1×3 submatrices and the 9×9 matrix associated with the fourth-rank tensors is subdivided into nine 3×3 submatrices, as shown in Section 1.1.4.9.1. The symmetry of the second-rank tensors means that submatrices **2** and **3** which are associated with them are equal.

Let us first consider the reduction of the tensor of elastic compliances. As in the case of the piezoelectric tensor, equation (1.1.4.9) can be written

$$S_{ij} = \sum_l s_{ijll}T_{ll} + \sum_{k\neq l}(s_{ijkl} + s_{ijlk})T_{kl}. \qquad (1.1.4.11)$$

The sums $(s_{ijkl} + s_{ijlk})$ for $k \neq l$ have a definite physical meaning, but it is impossible to devise an experiment permitting s_{ijkl} and s_{ijlk} to be measured separately. It is therefore usual to set them equal in order to avoid an unnecessary constant:

$$s_{ijkl} = s_{ijlk}.$$