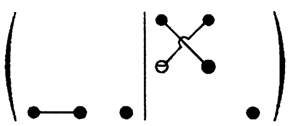
International Tables for Crystallography (2006). Vol. D, Section 1.1.4.10.5, pp. 26–27.

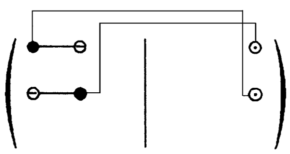
(iv) Groups $4/m$, $6/m$ and $(A_\infty/M)C$: all the components are equal to zero.

(v) Group $\bar{4}$:



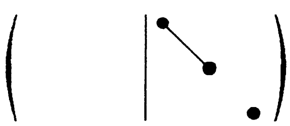
There are 4 independent components.

(vi) Group $\bar{6} = 3/m$:



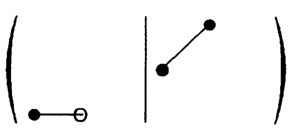
with the same conventions as for group 3. There are 2 independent components.

(vii) Group $\bar{4}2m$ – twofold axis parallel to Ox_1:



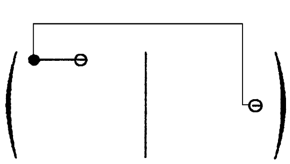
There are 2 independent components.

(viii) Group $\bar{4}2m$ – mirror perpendicular to Ox_1 (twofold axis at 45°):



The number of independent components is of course the same.

(ix) Group $\bar{6}2/m$:

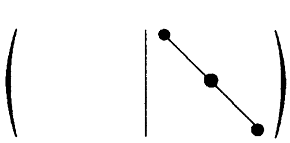


with the same conventions as for group 3. There is 1 independent component.

(x) Groups $4/mm$, $6/mm$ and $(A_\infty/M)\infty(A_2/M)C$: all the components are equal to zero.

1.1.4.10.4.6. *Cubic and spherical systems*

(i) Groups 23 and $\bar{4}3m$:



There is 1 independent component.

(ii) Groups 432 and ∞A_∞: it was seen in Section 1.1.4.8.6 that we have in this case

$$d_{123} = -d_{132}.$$

It follows that $d_{14} = 0$, all the components are equal to zero.

(iii) Groups $m3$, $m\bar{3}m$ and $\infty(A_\infty/M)C$: all the components are equal to zero.

1.1.4.10.5. *Reduction of the number of independent components of fourth-rank polar tensors due to the symmetry of the strain and stress tensors*

Let us consider five examples of fourth-rank tensors:

(i) *Elastic compliances*, s_{ijkl}, relating the resulting strain tensor S_{ij} to an applied stress T_{ij} (see Section 1.3.3.2):

$$S_{ij} = s_{ijkl} T_{kl}, \qquad (1.1.4.9)$$

where the compliances s_{ijkl} are the components of a tensor of rank 4.

(ii) *Elastic stiffnesses*, c_{ijkl} (see Section 1.3.3.2):

$$T_{ij} = c_{ijkl} S_{kl}.$$

(iii) *Piezo-optic coefficients*, π_{ijkl}, relating the variation $\Delta\eta_{ij}$ of the dielectric impermeability to an applied stress T_{kl} (*photoelastic effect* – see Section 1.6.7):

$$\Delta\eta_{ij} = \pi_{ijkl} T_{kl}.$$

(iv) *Elasto-optic coefficients*, p_{ijkl}, relating the variation $\Delta\eta_{ij}$ of the dielectric impermeability to the strain S_{kl}:

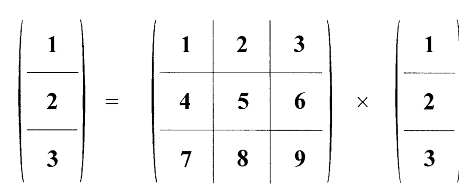
$$\Delta\eta_{ij} = p_{ijkl} S_{kl}.$$

(v) *Electrostriction coefficients*, Q_{ijkl}, which appear in equation (1.1.4.4):

$$S_{ij} = Q_{ijkl} E_k E_l, \qquad (1.1.4.10)$$

where only the second-order terms are considered.

In each of the equations from (1.1.4.9) to (1.1.4.10), the contracted product of a fourth-rank tensor by a symmetric second-rank tensor is equal to a symmetric second-rank tensor. As in the case of the third-rank tensors, this results in a reduction of the number of independent components, but because of the properties of the strain Voigt matrix, and because two of the tensors are endowed with intrinsic symmetry (the elastic tensors), the reduction is different for each of the five tensors. The above relations can be written in matrix form:



where the second-rank tensors are represented by 1×9 column matrices, which can each be subdivided into three 1×3 submatrices and the 9×9 matrix associated with the fourth-rank tensors is subdivided into nine 3×3 submatrices, as shown in Section 1.1.4.9.1. The symmetry of the second-rank tensors means that submatrices **2** and **3** which are associated with them are equal.

Let us first consider the reduction of the tensor of elastic compliances. As in the case of the piezoelectric tensor, equation (1.1.4.9) can be written

$$S_{ij} = \sum_l s_{ijll} T_{ll} + \sum_{k \neq l} (s_{ijkl} + s_{ijlk}) T_{kl}. \qquad (1.1.4.11)$$

The sums $(s_{ijkl} + s_{ijlk})$ for $k \neq l$ have a definite physical meaning, but it is impossible to devise an experiment permitting s_{ijkl} and s_{ijlk} to be measured separately. It is therefore usual to set them equal in order to avoid an unnecessary constant:

$$s_{ijkl} = s_{ijlk}.$$

Furthermore, the left-hand term of (1.1.4.11) remains unchanged if we interchange the indices i and j. The terms on the right-hand side therefore also remain unchanged, whatever the value of T_{ll} or T_{kl}. It follows that

$$s_{ijll} = s_{jill}$$
$$s_{ijkl} = s_{ijlk} = s_{jikl} = s_{jilk}.$$

Similar relations hold for c_{ijkl}, Q_{ijkl}, p_{ijkl} and π_{ijkl}: the submatrices **2** and **3**, **4** and **7**, **5**, **6**, **8** and **9**, respectively, are equal.

Equation (1.4.1.11) can be rewritten, introducing the coefficients of the Voigt strain matrix:

$$S_\alpha = S_{ii} = \sum_l s_{iill} T_{ll} + \sum_{k \neq l} (s_{iikl} + s_{iilk}) T_{kl} \quad (\alpha = 1, 2, 3)$$
$$S_\alpha = S_{ij} + S_{ji} = \sum_l (s_{ijll} + s_{jill}) T_{ll} + \sum_{k \neq l} (s_{ijkl} + s_{ijlk} + s_{jikl} + s_{jilk}) T_{kl} \quad (\alpha = 4, 5, 6).$$

We shall now introduce a two-index notation for the elastic compliances, according to the following conventions:

$$\left.\begin{array}{lll} i = j; & k = l; & s_{\alpha\beta} = s_{iill} \\ i = j; & k \neq l; & s_{\alpha\beta} = s_{iikl} + s_{iilk} \\ i \neq j; & k = l; & s_{\alpha\beta} = s_{ijkk} + s_{jikk} \\ i \neq j; & k \neq l; & s_{\alpha\beta} = s_{ijkl} + s_{jikl} + s_{ijlk} + s_{jilk}. \end{array}\right\} \quad (1.1.4.12)$$

We have thus associated with the fourth-rank tensor a square 6×6 matrix with 36 coefficients:

α \ β	1	2	3	4	5	6
1	11	12	13	14	15	16
2	21	22	23	24	25	26
3	31	32	33	34	35	36
4	41	42	43	44	45	46
5	51	52	53	54	55	56
6	61	62	63	64	65	66

One can translate relation (1.1.4.12) using the 9×9 matrix representing s_{ijkl} by adding term by term the coefficients of submatrices **2** and **3**, **4** and **7** and **5**, **6**, **8** and **9**, respectively:

$$\left(\begin{array}{c}\mathbf{1}\\ \hline \mathbf{2+3}\end{array}\right) = \left(\begin{array}{c|c}\mathbf{1} & \mathbf{2}\\ \hline \mathbf{4+7} & \mathbf{5+6+8+9}\end{array}\right) \times \left(\begin{array}{c}\mathbf{1}\\ \hline \mathbf{2+3}\end{array}\right)$$

Using the two-index notation, equation (1.1.4.9) becomes

$$S_\alpha = s_{\alpha\beta} T_\beta. \quad (1.1.4.13)$$

A similar development can be applied to the other fourth-rank tensors π_{ijkl}, which will be replaced by 6×6 matrices with 36 coefficients, according to the following rules.

(i) *Elastic stiffnesses*, c_{ijkl} and *elasto-optic coefficients*, p_{ijkl}:

$$\left(\begin{array}{c}\mathbf{1}\\ \hline \mathbf{2}\end{array}\right) = \left(\begin{array}{c|c}\mathbf{1} & \mathbf{2}\\ \hline \mathbf{4} & \mathbf{5}\end{array}\right) \times \left(\begin{array}{c}\mathbf{1}\\ \hline \mathbf{2}\end{array}\right)$$

where

$$c_{\alpha\beta} = c_{ijkl}$$
$$p_{\alpha\beta} = p_{ijkl}.$$

(ii) *Piezo-optic coefficients*, π_{ijkl}:

$$\left(\begin{array}{c}\mathbf{1}\\ \hline \mathbf{2}\end{array}\right) = \left(\begin{array}{c|c}\mathbf{1} & \mathbf{2+3}\\ \hline \mathbf{4} & \mathbf{5+6}\end{array}\right) \times \left(\begin{array}{c}\mathbf{1}\\ \hline \mathbf{2}\end{array}\right)$$

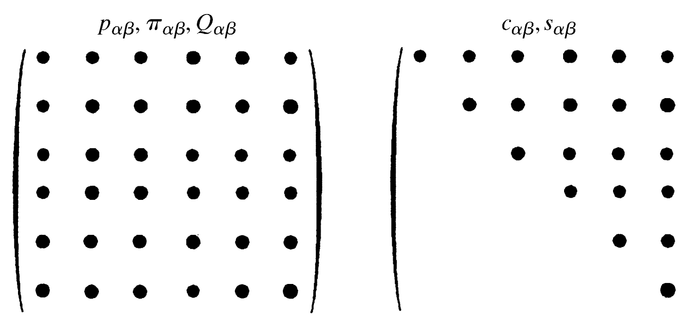
where

$$\left.\begin{array}{lll} i = j; & k = l; & \pi_{\alpha\beta} = \pi_{iill} \\ i = j; & k \neq l; & \pi_{\alpha\beta} = \pi_{iikl} + \pi_{iilk} \\ i \neq j; & k = l; & \pi_{\alpha\beta} = \pi_{ijkk} = \pi_{jikk} \\ i \neq j; & k \neq l; & \pi_{\alpha\beta} = \pi_{ijkl} + \pi_{jikl} = \pi_{ijlk} + \pi_{jilk}. \end{array}\right\}$$

(iii) *Electrostriction coefficients*, Q_{ijkl}: same relation as for the elastic compliances.

1.1.4.10.6. *Independent components of the matrix associated with a fourth-rank tensor according to the following point groups*

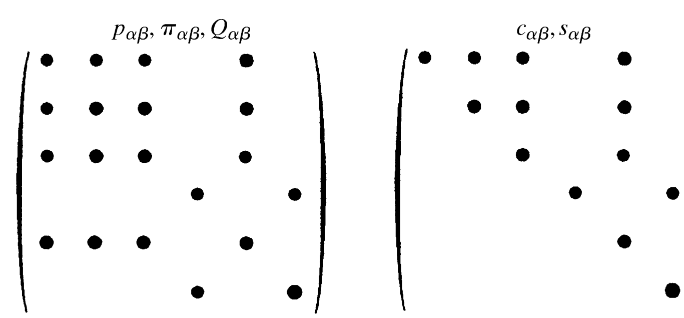
1.1.4.10.6.1. *Triclinic system, groups* $\bar{1}$, 1



36 independent components 21 independent components

1.1.4.10.6.2. *Monoclinic system*

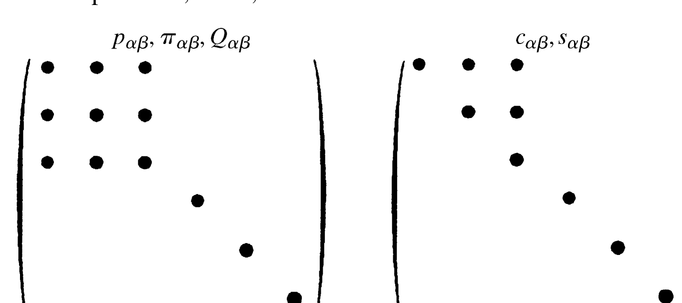
Groups $2/m$, 2, m, twofold axis parallel to Ox_2:



20 independent components 13 independent components

1.1.4.10.6.3. *Orthorhombic system*

Groups mmm, $2mm$, 222:



12 independent components 9 independent components