

1.1. INTRODUCTION TO THE PROPERTIES OF TENSORS

1.1.4.1. Introduction – Neumann’s principle

We saw in Section 1.1.1 that physical properties express in general the response of a medium to an impetus. It has been known for a long time that symmetry considerations play an important role in the study of physical phenomena. These considerations are often very fruitful and have led, for instance, to the discovery of piezoelectricity by the Curie brothers in 1880 (Curie & Curie, 1880, 1881). It is not unusual for physical properties to be related to asymmetries. This is the case in electrical polarization, optical activity *etc.* The first to codify this role was the German physicist and crystallographer F. E. Neumann, who expressed in 1833 the *symmetry principle*, now called *Neumann’s principle*: *if a crystal is invariant with respect to certain symmetry elements, any of its physical properties must also be invariant with respect to the same symmetry elements* (Neumann, 1885).

This principle may be illustrated by considering the optical properties of a crystal. In an anisotropic medium, the index of refraction depends on direction. For a given wave normal, two waves may propagate, with different velocities; this is the double refraction effect. The indices of refraction of the two waves vary with direction and can be found by using the index ellipsoid known as the *optical indicatrix* (see Section 1.6.3.2). Consider the central section of the ellipsoid perpendicular to the direction of propagation of the wave. It is an ellipse. The indices of the two waves that may propagate along this direction are equal to the semi-axes of that ellipse. There are two directions for which the central section is circular, and therefore two wave directions for which there is no double refraction. These directions are called optic axes, and the medium is said to be *biaxial*. If the medium is invariant with respect to a threefold, a fourfold or a sixfold axis (as in a trigonal, tetragonal or hexagonal crystal, for instance), its ellipsoid must also be invariant with respect to the same axis, according to Neumann’s principle. As an ellipsoid can only be ordinary or of revolution, the indicatrix of a trigonal, tetragonal or hexagonal crystal is necessarily an ellipsoid of revolution that has only one circular central section and one optic axis. These crystals are said to be *uniaxial*. In a cubic crystal that has four threefold axes, the indicatrix must have several axes of revolution, it is therefore a sphere, and cubic media behave as isotropic media for properties represented by a tensor of rank 2.

1.1.4.2. Curie laws

The example given above shows that the symmetry of the property may possess a higher symmetry than the medium. The property is represented in that case by the indicatrix. The symmetry of an ellipsoid is

$$\frac{A_2}{M} \frac{A'_2}{M'} \frac{A''_2}{M''} C = mmm \text{ for any ellipsoid}$$

(orthorhombic symmetry)

$$\frac{A_\infty}{M} \frac{\infty A_2}{\infty M} C = \frac{\infty}{m} m \text{ for an ellipsoid of revolution}$$

(cylindrical symmetry)

$$\infty \frac{A_\infty}{M} C = \infty \frac{\infty}{m} \text{ for a sphere}$$

(spherical symmetry).

[Axes A_∞ are axes of revolution, or *axes of isotropy*, introduced by Curie (1884, 1894), cf. *International Tables for Crystallography* (2002), Vol. A, Table 10.1.4.2.]

The symmetry of the indicatrix is identical to that of the medium if the crystal belongs to the orthorhombic holohedry and is higher in all other cases.

This remark is the basis of the generalization of the symmetry principle by P. Curie (1859–1906). He stated that (Curie, 1894):

(i) *the symmetry characteristic of a phenomenon is the highest compatible with the existence of the phenomenon;*

(ii) *the phenomenon may exist in a medium that possesses that symmetry or that of a subgroup of that symmetry;*

and concludes that some symmetry elements may coexist with the phenomenon but that their presence is not necessary. On the contrary, what is necessary is the *absence* of certain symmetry elements: ‘asymmetry creates the phenomenon’ (*‘C’est la dissymétrie qui crée le phénomène’*; Curie, 1894, p. 400). Noting that physical phenomena usually express relations between a cause and an effect (an influence and a response), P. Curie restated the two above propositions in the following way, now known as Curie laws, although they are not, properly speaking, laws:

- (i) *the asymmetry of the effects must pre-exist in the causes;*
- (ii) *the effects may be more symmetric than the causes.*

The application of the Curie laws enable one to determine the symmetry characteristic of a phenomenon. Let us consider the phenomenon first as an effect. If Φ is the symmetry of the phenomenon and C the symmetry of the cause that produces it,

$$C \leq \Phi.$$

Let us now consider the phenomenon as a cause producing a certain effect with symmetry E :

$$\Phi \leq E.$$

We can therefore conclude that

$$C \leq \Phi \leq E.$$

If we choose among the various possible causes the most symmetric one, and among the various possible effects the one with the lowest symmetry, we can then determine the symmetry that characterizes the phenomenon.

As an example, let us determine the symmetry associated with a mechanical force. A force can be considered as the result of a traction effort, the symmetry of which is $A_\infty \infty M$. If considered as a cause, its effect may be the motion of a sphere in a given direction (for example, a spherical ball falling under its own weight). Again, the symmetry is $A_\infty \infty M$. The symmetries associated with the force considered as a cause and as an effect being the same, we may conclude that $A_\infty \infty M$ is its characteristic symmetry.

1.1.4.3. Symmetries associated with an electric field and with magnetic induction (flux density)

1.1.4.3.1. Symmetry of an electric field

Considered as an effect, an electric field may have been produced by two circular coaxial electrodes, the first one carrying positive electric charges, the other one negative charges (Fig. 1.1.4.1). The cause possesses an axis of revolution and an infinity of mirrors parallel to it, $A_\infty \infty M$. Considered as a cause, the electric field induces for instance the motion of a spherical electric charge parallel to itself. The associated symmetry is the same in each case, and the symmetry of the electric field is identical to that of a force, $A_\infty \infty M$. The electric polarization or the electric displacement have the same symmetry.

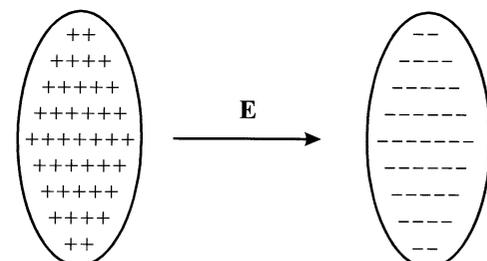


Fig. 1.1.4.1. Symmetry of an electric field.

1. TENSORIAL ASPECTS OF PHYSICAL PROPERTIES

1.1.4.3.2. Symmetry of magnetic induction

The determination of the symmetry of magnetic quantities is more delicate. Considered as an effect, magnetic induction may be obtained by passing an electric current in a loop (Fig. 1.1.4.2). The corresponding symmetry is that of a cylinder rotating around its axis, $(A_\infty/M)C$. Conversely, the variation of the flux of magnetic induction through a loop induces an electric current in the loop. If the magnetic induction is considered as a cause, its effect has the same symmetry. The symmetry associated with the magnetic induction is therefore $(A_\infty/M)C$.

This symmetry is completely different from that of the electric field. This difference can be understood by reference to Maxwell's equations, which relate electric and magnetic quantities:

$$\text{curl } \mathbf{E} = \nabla \wedge \mathbf{E} = -\frac{\partial \mathbf{B}}{\partial t}; \quad \text{curl } \mathbf{H} = \nabla \wedge \mathbf{H} = \frac{\partial \mathbf{D}}{\partial t}.$$

It was seen in Section 1.1.3.8.3 that the curl is an axial vector because it is a vector product. Maxwell's equations thus show that if the electric quantities (\mathbf{E} , \mathbf{D}) are polar vectors, the magnetic quantities (\mathbf{B} , \mathbf{H}) are axial vectors and *vice versa*; the equations of Maxwell are, in effect, perfectly symmetrical on this point. Indeed, one could have been tempted to determine the symmetry of the magnetic field by considering interactions between magnets, which would have led to the symmetry $A_\infty \infty M$ for the magnetic quantities. However, in the world where we live and where the origin of magnetism is in the spin of the electron, the magnetic field is an axial vector of symmetry $(A_\infty/M)C$ while the electric field is a polar vector of symmetry $A_\infty \infty M$.

1.1.4.4. Superposition of several causes in the same medium – pyroelectricity and piezoelectricity

1.1.4.4.1. Introduction

Let us now consider a phenomenon resulting from the superposition of several causes in the same medium. The symmetry of the global cause is the intersection of the groups of symmetry of the various causes: the asymmetries add up (Curie, 1894). This remark can be applied to the determination of the point groups where physical properties such as pyroelectricity or piezoelectricity are possible.

1.1.4.4.2. Pyroelectricity

Pyroelectricity is the property presented by certain materials that exhibit electric polarization when the temperature is changed uniformly. Actually, this property appears in crystals for which the centres of gravity of the positive and negative charges do not coincide in the unit cell. They present therefore a spontaneous polarization that varies with temperature because, owing to thermal expansion, the distances between these centres of gravity are temperature dependent. A very important case is that of the ferroelectric crystals where the direction of the polarization can be changed under the application of an external electric field.

From the viewpoint of symmetry, pyroelectricity can be considered as the superposition of two causes, namely the crystal

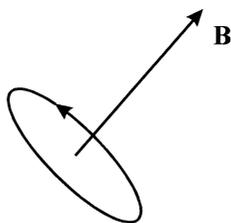


Fig. 1.1.4.2. Symmetry of magnetic induction.

with its symmetry on one hand and the increase of temperature, which is isotropic, on the other. The intersection of the groups of symmetry of the two causes is in this case identical to the group of symmetry of the crystal. The symmetry associated with the effect is that of the electric polarization that is produced, $A_\infty \infty M$. Since the asymmetry of the cause must pre-exist in the causes, the latter may not possess more than one axis of symmetry nor mirrors other than those parallel to the single axis. The only crystal point groups compatible with this condition are

$$1, 2, 3, 4, 6, m, 2mm, 3m, 4mm, 6mm.$$

There are therefore only ten crystallographic groups that are compatible with the pyroelectric effect. For instance, tourmaline, in which the effect was first observed, belongs to $3m$.

1.1.4.4.3. Piezoelectricity

Piezoelectricity, discovered by the Curie brothers (Curie & Curie, 1880), is the property presented by certain materials that exhibit an electric polarization when submitted to an applied mechanical stress such as a uniaxial compression (see, for instance, Cady, 1964; Ikeda, 1990). Conversely, their shape changes when they are submitted to an external electric field; this is the converse piezoelectric effect. The physical interpretation of piezoelectricity is the following: under the action of the applied stress, the centres of gravity of negative and positive charges move to different positions in the unit cell, which produces an electric polarization.

From the viewpoint of symmetry, piezoelectricity can be considered as the superposition of two causes, the crystal with its own symmetry and the applied stress. The symmetry associated with a uniaxial compression is that of two equal and opposite forces, namely $A_\infty/M \infty A_2/\infty MC$. The effect is an electric polarization, of symmetry $A_\infty \infty M$, which must be higher than or equal to the intersection of the symmetries of the two causes:

$$\frac{A_\infty \infty A_2}{M \infty M} C \cap S_{\text{crystal}} \leq A_\infty \infty M,$$

where S_{crystal} denotes the symmetry of the crystal.

It may be noted that the effect does not possess a centre of symmetry. The crystal point groups compatible with the property of piezoelectricity are therefore among the 21 noncentrosymmetric point groups. More elaborate symmetry considerations show further that group 432 is also not compatible with piezoelectricity. This will be proved in Section 1.1.4.10.4 using the symmetry properties of tensors. There are therefore 20 point groups compatible with piezoelectricity:

$$1, 2, m, 222, 2mm, \\ 3, 32, 3m, 4, \bar{4}, 422, 4mm, \bar{4}2m, 6, \bar{6}, 622, 6mm, \bar{6}2m \\ 23, \bar{4}3m.$$

The intersection of the symmetries of the crystal and of the applied stress depend of course on the orientation of this stress relative to the crystallographic axes. Let us take, for instance, a crystal of quartz, which belongs to group $32 = A_33A_2$. The above condition becomes

$$\frac{A_\infty \infty A_2}{M \infty M} C \cap A_33A_2 \leq A_\infty \infty M.$$

If the applied compression is parallel to the threefold axis, the intersection is identical to the symmetry of the crystal, A_33A_2 , which possesses symmetry elements that do not exist in the effect, and piezoelectricity cannot appear. This is of course obvious because the threefold axis is not polar. For all other directions, piezoelectricity may appear.