*International Tables for Crystallography* (2006). Vol. D, Section 1.1.4.9.9.7, pp. 23–24.



(ii) Groups $\bar{3}m$, 32, $3m$

| *ij* \ *kl* | 11 | 22 | 33 | 23 | 31 | 12 | 32 | 13 | 21 |
|---|---|---|---|---|---|---|---|---|---|
| 11 | 1111 | 1122 | 1133 | 1123 | | | 1132 | | |
| 22 | | 1111 | 1133 | −1123 | | | −1132 | | |
| 33 | | | 3333 | | | | | | |
| 23 | | | | 2323 | | | 2332 | | |
| 31 | | | | | 3131 | 1132 | | 2332 | 1132 |
| 12 | | | | | | 1212 | | 1123 | 1221 |
| 32 | | | | | | | 3131 | | |
| 13 | | | | | | | | 2323 | 1123 |
| 21 | | | | | | | | | 1212 |

with

$$t_{1111} - t_{1122} = t_{1212} + t_{1221}.$$

There are 11 independent components.

1.1.4.9.9.5. *Tetragonal system*

(i) Groups $4/m$, 4, $\bar{4}$

| *ij* \ *kl* | 11 | 22 | 33 | 23 | 31 | 12 | 32 | 13 | 21 |
|---|---|---|---|---|---|---|---|---|---|
| 11 | 1111 | 1122 | 1133 | | | 1112 | | | −2212 |
| 22 | | 1111 | 1133 | | | 2212 | | | −1112 |
| 33 | | | 3333 | | | 3312 | | | −3312 |
| 23 | | | | 2323 | 2331 | | 2332 | | |
| 31 | | | | | 3131 | | | 2332 | |
| 12 | | | | | | 1212 | | | 1221 |
| 32 | | | | | | | 3131 | −2331 | |
| 13 | | | | | | | | 2323 | |
| 21 | | | | | | | | | 1212 |

There are 13 independent components.

(ii) Groups $4/mm$, 422, $4mm$, $\bar{4}2m$

| *ij* \ *kl* | 11 | 22 | 33 | 23 | 31 | 12 | 32 | 13 | 21 |
|---|---|---|---|---|---|---|---|---|---|
| 11 | 1111 | 1122 | 1133 | | | | | | |
| 22 | | 1111 | 1133 | | | | | | |
| 33 | | | 3333 | | | | | | |
| 23 | | | | 2323 | | | 2332 | | |
| 31 | | | | | 3131 | | | 3113 | |
| 12 | | | | | | 1212 | | | 1221 |
| 32 | | | | | | | 3131 | | |
| 13 | | | | | | | | 2323 | |
| 21 | | | | | | | | | 1212 |

There are 9 independent components.

1.1.4.9.9.6. *Hexagonal and cylindrical systems*

(i) Groups $6/m$, $\bar{6}$, 6; $(A_\infty/M)C$, $A_\infty$

| *ij* \ *kl* | 11 | 22 | 33 | 23 | 31 | 12 | 32 | 13 | 21 |
|---|---|---|---|---|---|---|---|---|---|
| 11 | 1111 | 1122 | 1133 | | | 1112 | | | 1121 |
| 22 | | 1111 | 1133 | | | −1121 | | | −1112 |
| 33 | | | 3333 | | | 3312 | | | −3312 |
| 23 | | | | 2323 | 2331 | | 2332 | | |
| 31 | | | | | 3131 | | 3132 | 2332 | |
| 12 | | | | | | 1212 | | | 1221 |
| 32 | | | | | | | 3131 | −2331 | |
| 13 | | | | | | | | 2323 | |
| 21 | | | | | | | | | 1212 |

with

$$t_{1111} - t_{1122} = t_{1212} + t_{1221}.$$

There are 12 independent components.

(ii) Groups $6/mm$, 622, $6mm$, $\bar{6}2m$; $(A_\infty/M)\infty(A_2/M)C$, $A_\infty \infty A_2$

| *ij* \ *kl* | 11 | 22 | 33 | 23 | 31 | 12 | 32 | 13 | 21 |
|---|---|---|---|---|---|---|---|---|---|
| 11 | 1111 | 1122 | 1133 | | | | | | |
| 22 | | 1111 | 1133 | | | | | | |
| 33 | | | 3333 | | | | | | |
| 23 | | | | 2323 | | | 2332 | | |
| 31 | | | | | 3131 | | | 3113 | |
| 12 | | | | | | 1212 | | | 1221 |
| 32 | | | | | | | 3131 | | |
| 13 | | | | | | | | 2323 | |
| 21 | | | | | | | | | 1212 |

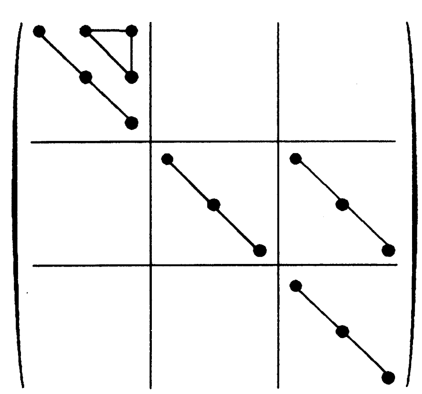
with

$$t_{1111} - t_{1122} = t_{1212} + t_{1221}.$$

There are 10 independent components.

1.1.4.9.9.7. *Cubic system*

(i) Groups 23, $\bar{3}m$



with

$$t_{1111} - t_{1122} = t_{1212} + t_{1221}.$$

There are 5 independent components.

(ii) Groups $m\bar{3}m$, 432, $\bar{4}3m$, and spherical system: the reduced tensors are already symmetric (see Sections 1.1.4.9.7 and 1.1.4.9.8).

### 1.1.4.10. *Reduced form of polar and axial tensors – matrix representation*

#### 1.1.4.10.1. *Introduction*

Many tensors representing physical properties or physical quantities appear in relations involving symmetric tensors. Consider, for instance, the strain $S_{ij}$ resulting from the application of an electric field $\mathbf{E}$ (the piezoelectric effect):

$$S_{ij} = d_{ijk}E_k + Q_{ijkl}E_kE_l, \qquad (1.1.4.4)$$

where the first-order terms $d_{ijk}$ represent the components of the third-rank converse piezoelectric tensor and the second-order terms $Q_{ijkl}$ represent the components of the fourth-rank electrostriction tensor. In a similar way, the direct piezoelectric effect corresponds to the appearance of an electric polarization $\mathbf{P}$ when a stress $T_{jk}$ is applied to a crystal:

$$P_i = d_{ijk}T_{jk}. \qquad (1.1.4.5)$$

Owing to the symmetry properties of the strain and stress tensors (see Sections 1.3.1 and 1.3.2) and of the tensor product $E_kE_l$, there occurs a further reduction of the number of independent components of the tensors which are engaged in a contracted product with them, as is shown in Section 1.1.4.10.3 for third-rank tensors and in Section 1.1.4.10.5 for fourth-rank tensors.

#### 1.1.4.10.2. *Stress and strain tensors – Voigt matrices*

The stress and strain tensors are symmetric because body torques and rotations are not taken into account, respectively (see Sections 1.3.1 and 1.3.2). Their components are usually represented using Voigt's one-index notation.

(i) Strain tensor

$$\left.\begin{array}{lll} S_1 = S_{11}; & S_2 = S_{22}; & S_3 = S_{33}; \\ S_4 = S_{23} + S_{32}; & S_5 = S_{31} + S_{13}; & S_6 = S_{12} + S_{21}; \\ S_4 = 2S_{23} = 2S_{32}; & S_5 = 2S_{31} = 2S_{13}; & S_6 = 2S_{12} = 2S_{21}. \end{array}\right\} \qquad (1.1.4.6)$$

The Voigt components $S_\alpha$ form a Voigt matrix:

$$\begin{pmatrix} S_1 & S_6 & S_5 \\ & S_2 & S_4 \\ & & S_3 \end{pmatrix}.$$

The terms of the leading diagonal represent the elongations (see Section 1.3.1). It is important to note that the non-diagonal terms, which represent the shears, are here equal to *twice* the corresponding components of the strain tensor. The components $S_\alpha$ of the Voigt strain matrix are therefore *not* the components of a tensor.

(ii) Stress tensor

$$\left.\begin{array}{lll} T_1 = T_{11}; & T_2 = T_{22}; & T_3 = T_{33}; \\ T_4 = T_{23} = T_{32}; & T_5 = T_{31} = T_{13}; & T_6 = T_{12} = T_{21}. \end{array}\right\}$$

The Voigt components $T_\alpha$ form a Voigt matrix:

$$\begin{pmatrix} T_1 & T_6 & T_5 \\ & T_2 & T_4 \\ & & T_3 \end{pmatrix}.$$

The terms of the leading diagonal correspond to principal normal constraints and the non-diagonal terms to shears (see Section 1.3.2).

#### 1.1.4.10.3. *Reduction of the number of independent components of third-rank polar tensors due to the symmetry of the strain and stress tensors*

Equation (1.1.4.5) can be written

$$P_i = \sum_j d_{ijj}T_{jj} + \sum_{j\neq k}(d_{ijk} + d_{ikj})T_{jk}.$$

The sums $(d_{ijk} + d_{ikj})$ for $j \neq k$ have a definite physical meaning, but it is impossible to devise an experiment that permits $d_{ijk}$ and $d_{ikj}$ to be measured separately. It is therefore usual to set them equal:

$$d_{ijk} = d_{ikj}. \qquad (1.1.4.7)$$

It was seen in Section 1.1.4.8.1 that the components of a third-rank tensor can be represented as a $9 \times 3$ matrix which can be subdivided into three $3 \times 3$ submatrices:

$$(\;\mathbf{1}\;|\;\mathbf{2}\;|\;\mathbf{3}\;).$$

Relation (1.1.4.7) shows that submatrices **1** and **2** are identical. One puts, introducing a two-index notation,

$$\left.\begin{array}{rl} d_{ijj} = d_{i\alpha} & (\alpha = 1, 2, 3) \\ d_{ijk} + d_{ikj}\ (j \neq k) = d_{i\alpha} & (\alpha = 4, 5, 6). \end{array}\right\}$$

Relation (1.1.4.7) becomes

$$P_i = d_{i\alpha}T_\alpha.$$

The coefficients $d_{i\alpha}$ may be written as a $3 \times 6$ matrix:

$$\left(\begin{array}{ccc|ccc} 11 & 12 & 13 & 14 & 15 & 16 \\ 21 & 22 & 23 & 24 & 25 & 26 \\ 31 & 32 & 33 & 34 & 35 & 36 \end{array}\right).$$

This matrix is constituted by two $3 \times 3$ submatrices. The left-hand one is identical to the submatrix **1**, and the right-hand one is equal to the sum of the two submatrices **2** and **3**:

$$(\;\mathbf{1}\;|\;\mathbf{2+3}\;).$$

The inverse piezoelectric effect expresses the strain in a crystal submitted to an applied electric field:

$$S_{ij} = d_{ijk}E_k,$$

where the matrix associated with the coefficients $d_{ijk}$ is a $9 \times 3$ matrix which is the transpose of that of the coefficients used in equation (1.1.4.5), as shown in Section 1.1.1.4.

The components of the Voigt strain matrix $S_\alpha$ are then given by

$$\left.\begin{array}{ll} S_\alpha = d_{iik}E_k & (\alpha = 1, 2, 3) \\ S_\alpha = S_{ij} + S_{ji} = (d_{ijk} + d_{jik})E_k & (\alpha = 4, 5, 6). \end{array}\right\}$$

This relation can be written simply as

$$S_\alpha = d_{\alpha k}E_k,$$

where the matrix of the coefficients $d_{\alpha k}$ is a $6 \times 3$ matrix which is the transpose of the $d_{i\alpha}$ matrix.

There is another set of piezoelectric constants (see Section 1.1.5) which relates the stress, $T_{ij}$, and the electric field, $E_k$, which are both intensive parameters:

$$T_{ij} = e_{ijk}E_k, \qquad (1.1.4.8)$$

where a new piezoelectric tensor is introduced, $e_{ijk}$. Its components can be represented as a $3 \times 9$ matrix: