

1. TENSORIAL ASPECTS OF PHYSICAL PROPERTIES

1.1.5.2. Other forms of the piezoelectric constants

We use here another Gibbs function, the electric Gibbs function, \mathcal{G}_2 , defined by

$$\mathcal{G}_2 = \mathcal{U} - E_n D_n - \Theta \sigma.$$

Differentiation of \mathcal{G} gives

$$d\mathcal{G}_2 = -D_n dE_n + T_{ij} dS_{ij} - \sigma d\Theta.$$

It follows that

$$T_{ij} = \frac{\partial \mathcal{G}_2}{\partial S_{ij}}; \quad D_n = -\frac{\partial \mathcal{G}_2}{\partial E_n}; \quad \sigma = -\frac{\partial \mathcal{G}_2}{\partial \Theta}$$

and a set of relations analogous to (1.1.5.1):

$$\left. \begin{aligned} T_{ij} &= (c_{ijkl})^{E,\Theta} S_{kl} - (e_{ijn})^{S,\Theta} E_n - (\lambda_{ij})^{E,S} \delta\Theta \\ D_n &= (e_{nij})^{E,\Theta} S_{ij} + (\varepsilon_{nm})^{S,\Theta} E_m + (p_n)^S \delta\Theta \\ \delta\sigma &= (\lambda_{ij})^E S_{ij} + (p_n)^S E_n + \rho C^{E,S} \delta\Theta/\Theta, \end{aligned} \right\} \quad (1.1.5.2)$$

where the components $(c_{ijkl})^{E,\Theta}$ are the isothermal elastic stiffnesses at constant field and constant temperature,

$$e_{ijn} = -\frac{\partial T_{ij}}{\partial E_n} = -\frac{\partial^2 \mathcal{G}_2}{\partial E_n \partial S_{ij}} = -\frac{\partial^2 \mathcal{G}_2}{\partial S_{ij} \partial E_n} = \frac{\partial D_n}{\partial S_{ij}} = e_{nij}$$

are the piezoelectric stress coefficients at constant strain and constant temperature,

$$\lambda_{ij} = -\frac{\partial T_{ij}}{\partial \Theta} = -\frac{\partial^2 \mathcal{G}_2}{\partial \Theta \partial S_{ij}} = -\frac{\partial^2 \mathcal{G}_2}{\partial S_{ij} \partial \Theta} = \frac{\partial \delta\sigma}{\partial S_{ij}}$$

are the temperature-stress constants and

$$p_n = \frac{\partial D_n}{\partial \Theta} = -\frac{\partial^2 \mathcal{G}_2}{\partial \Theta \partial E_n} = -\frac{\partial^2 \mathcal{G}_2}{\partial E_n \partial \Theta} = \frac{\partial \delta\sigma}{\partial E_n}$$

are the components of the pyroelectric effect at constant strain.

The relations between these coefficients and the usual coefficients d_{kln} are easily obtained:

(i) At constant temperature and strain: if one puts $\delta\Theta = 0$ and $S_{kl} = 0$ in the first equation of (1.1.5.1) and (1.1.5.2), one obtains, respectively,

$$\begin{aligned} 0 &= s_{klj} T_{ij} + d_{kln} E_n \\ T_{ij} &= -e_{ijn} E_n, \end{aligned}$$

from which it follows that

$$d_{kln} = s_{klj} e_{ijn}$$

at constant temperature and strain.

(ii) At constant temperature and stress: if one puts $\delta\Theta = 0$ and $T_{ij} = 0$, one obtains in a similar way

$$\begin{aligned} S_{kl} &= d_{kln} E_n \\ 0 &= c_{ijkl} S_{kl} - e_{ijn} E_n, \end{aligned}$$

from which it follows that

$$e_{ijn} = c_{ijkl} d_{kln}$$

at constant temperature and stress.

1.1.5.3. Relation between the pyroelectric coefficients at constant stress and at constant strain

By combining relations (1.1.5.1) and (1.1.5.2), it is possible to obtain relations between the pyroelectric coefficients at constant stress, p_n^T , and the pyroelectric coefficients at constant strain, p_n^S , also called *real* pyroelectric coefficients, p_n^S . Let us put $T_{ij} = 0$ and

$E_n = 0$ in the first equation of (1.1.5.1). For a given variation of temperature, $\delta\Theta$, the observed strain is

$$S_{kl} = [\alpha_{kl}]^{E,T} \delta\Theta.$$

From the second equations of (1.1.5.1) and (1.1.5.2), it follows that

$$\begin{aligned} D_n &= p_n^T \delta\Theta \\ D_n &= e_{nkl} S_{kl} + p_n^S \delta\Theta. \end{aligned}$$

Substituting the expression S_{kl} and eliminating D_n , it follows that

$$p_n^T = e_{nkl} [\alpha_{kl}]^{E,T} + p_n^S. \quad (1.1.5.3)$$

This relation shows that part of the pyroelectric effect is actually due to the piezoelectric effect.

1.1.5.4. Adiabatic study

Piezoelectric resonators usually operate at a high frequency where there are no heat exchanges, and therefore in an adiabatic regime ($\Theta\delta\sigma = 0$). From the third equation of (1.1.5.1), we obtain a relation between the temperature variation, the applied stress and the electric field:

$$(\alpha_{ij})^E T_{ij} + (p_m)^T E_m + \frac{\rho C^{T,E}}{\Theta} \delta\Theta = 0.$$

If we substitute this relation in the two other relations of (1.1.5.1), we obtain two equivalent relations, but in the *adiabatic* regime:

$$\begin{aligned} S_{kl} &= (s_{klj})^{E,\sigma} T_{ij} + (d_{klm})^{T,\sigma} E_m \\ D_n &= (d_{nij})^{E,\sigma} T_{ij} + (\varepsilon_{nm})^{T,\sigma} E_m. \end{aligned}$$

By comparing these expressions with (1.1.5.1), we obtain the following relations between the adiabatic and the isothermal coefficients:

$$\begin{aligned} (s_{ijkl})^{E,\sigma} &= (s_{ijkl})^{E,\theta} - \frac{(\alpha_{ij})^E (\alpha_{kl})^E \Theta}{\rho C^{T,E}} \\ (d_{nij})^{E,\sigma} &= (d_{nij})^{E,\theta} - \frac{(p_n)^T (\alpha_{kl})^E \Theta}{\rho C^{T,E}} \\ (\varepsilon_{mn})^{T,\sigma} &= (\varepsilon_{mn})^{T,\theta} - \frac{(p_n)^T (p_m)^T \Theta}{\rho C^{T,E}}. \end{aligned}$$

1.1.6. Glossary

\mathbf{e}_i	basis vectors in direct space (covariant)
\mathbf{e}^i	basis vectors in reciprocal space (contravariant)
x^i	components of a vector in direct space (contravariant)
x_i	components of a vector in reciprocal space (covariant)
g_{ij}	components of the metric tensor
$\overset{j_1 \dots j_q}{i_1 \dots i_p}$	components of a tensor of rank n , p times covariant and q times contravariant ($n = p + q$)
A^T	transpose of matrix A
\otimes	tensor product
\wedge	outer product
\wedge	vector product
∂_i	partial derivative with respect to x_i
δ_i^j	Kronecker symbol
ε_{ijk}	permutation tensor
V	volume