whether the direction under consideration is outside or inside the asymptotic cone and intersects one or the other of the two hyperboloids.

Equally, one can connect the displacement vector $\mathbf{u}(\mathbf{r})$ directly with the quadric Q. Using the bilinear form

$$f(\mathbf{y}) = M_{ij} y_i y_j,$$

the gradient of $f(\mathbf{y})$, $\nabla(f)$, has as components

$$\partial f/\partial y^i = M_{ij} y_j = u_i.$$

One recognizes the components of the displacement vector $\mathbf{u}$, which is therefore parallel to the normal to the quadric Q at the extremity of the radius vector **Op** parallel to **r**.

The directions of the principal axes of Q correspond to the extremal values of y, *i.e.* to the stationary values (maximal or minimal) of the elongation. These values are the *principal elongations*.

If the deformation is a pure rotation

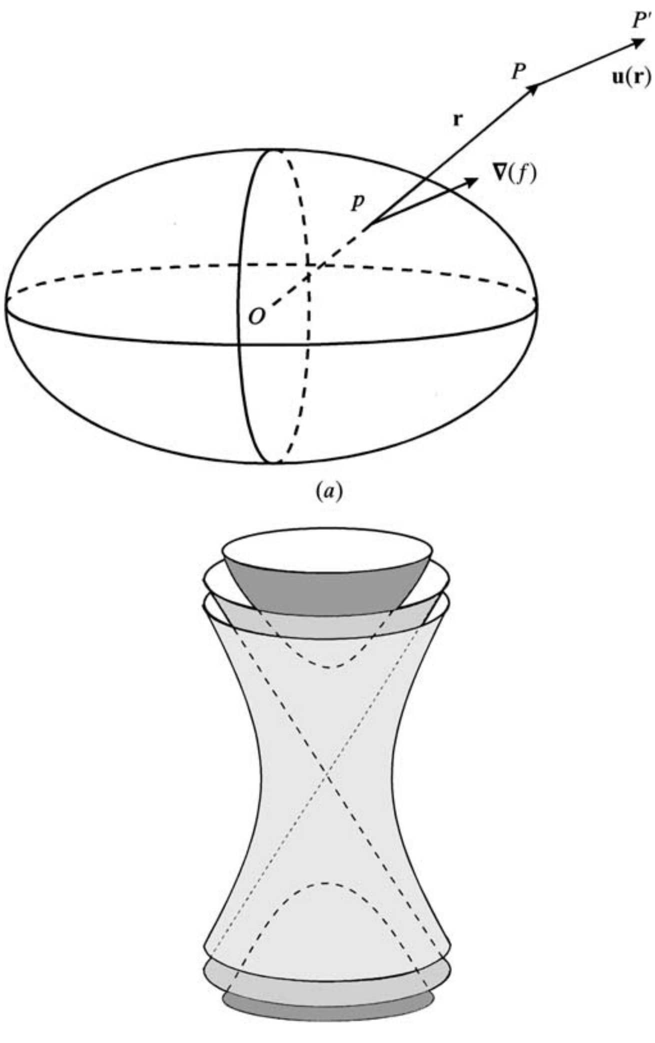

Fig. 1.3.1.3. Quadric of elongations. The displacement vector, $\mathbf{u}(\mathbf{r})$, at P in the deformed medium is parallel to the normal to the quadric at the intersection, p, of **OP** with the quadric. (*a*) The eigenvalues all have the same sign, the quadric is an ellipsoid. (*b*) The eigenvalues have mixed signs, the quadric is a hyperboloid with either one sheet (shaded in light grey) or two sheets (shaded in dark grey), depending on the sign of the constant ε [see equation (1.3.1.7)]; the cone asymptote is represented in medium grey. For a practical application, see Fig. 1.4.1.1.

$$B = \begin{pmatrix} \cos\theta & \sin\theta & 0 \\ -\sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{pmatrix},$$

$$M = \begin{pmatrix} \cos\theta - 1 & \sin\theta & 0 \\ -\sin\theta & \cos\theta - 1 & 0 \\ 0 & 0 & 0 \end{pmatrix}.$$

Hence we have

$$M_{ij} y_i y_j = (\cos\theta - 1)(y_1 - y_2) = \varepsilon.$$

The quadric Q is a cylinder of revolution having the axis of rotation as axis.

1.3.1.3. *Arbitrary but small deformations*

1.3.1.3.1. *Definition of the strain tensor*

If the deformation is small but arbitrary, *i.e.* if the products of two or more components of M_{ij} can be neglected with respect to unity, one can describe the deformation locally as a homogeneous asymptotic deformation. As was shown in Section 1.3.1.2.4, it can be put in the form of the product of a pure deformation corresponding to the symmetric part of M_{ij}, S_{ij}, and a pure rotation corresponding to the asymmetric part, A_{ij}:

$$\left.\begin{aligned} S_{ij} &= S_{ji} = \tfrac{1}{2}\left(\frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i}\right) \\ A_{ij} &= -A_{ji} = \tfrac{1}{2}\left(\frac{\partial u_i}{\partial x_j} - \frac{\partial u_j}{\partial x_i}\right). \end{aligned}\right\} \quad (1.3.1.8)$$

Matrix B can be written

$$B = I + A + S,$$

where I is the matrix identity. As the coefficients $\partial u_i/\partial x_j$ of M_{ij} are small, one can neglect the product $A \times S$ and one has

$$B = (I + A)(I + S).$$

$(I + S)$ is a symmetric matrix that represents a pure deformation. $(I + A)$ is an antisymmetric unitary matrix and, since A is small,

$$(I + A)^{-1} = (I - A).$$

Thus, $(I + A)$ represents a rotation. The axis of rotation is parallel to the vector with coordinates

$$\left.\begin{aligned} \Omega_1 &= \tfrac{1}{2}\left(\frac{\partial u_3}{\partial x_2} + \frac{\partial u_2}{\partial x_3}\right) = A_{32} \\ \Omega_2 &= \tfrac{1}{2}\left(\frac{\partial u_1}{\partial x_3} + \frac{\partial u_3}{\partial x_1}\right) = A_{13} \\ \Omega_3 &= \tfrac{1}{2}\left(\frac{\partial u_2}{\partial x_1} + \frac{\partial u_1}{\partial x_2}\right) = A_{21}, \end{aligned}\right\}$$

which is an eigenvector of $(I + A)$. The magnitude of the rotation is equal to the modulus of this vector.

In general, one is only interested in the pure deformation, *i.e.* in the form of the deformed object. Thus, one only wishes to know the quantities $(I + S)$ and the symmetric part of M. It is this symmetric part that is called the deformation tensor or the strain tensor. It is very convenient for applications to use the simplified notation due to Voigt:

$$S_1 = \frac{\partial u_1}{\partial x_1}; \quad S_2 = \frac{\partial u_2}{\partial x_2}; \quad S_3 = \frac{\partial u_3}{\partial x_3};$$
$$S_4 = \frac{\partial u_3}{\partial x_2} + \frac{\partial u_2}{\partial x_3}; \quad S_5 = \frac{\partial u_3}{\partial x_1} + \frac{\partial u_1}{\partial x_3}; \quad S_6 = \frac{\partial u_2}{\partial x_1} + \frac{\partial u_1}{\partial x_2}.$$

  references