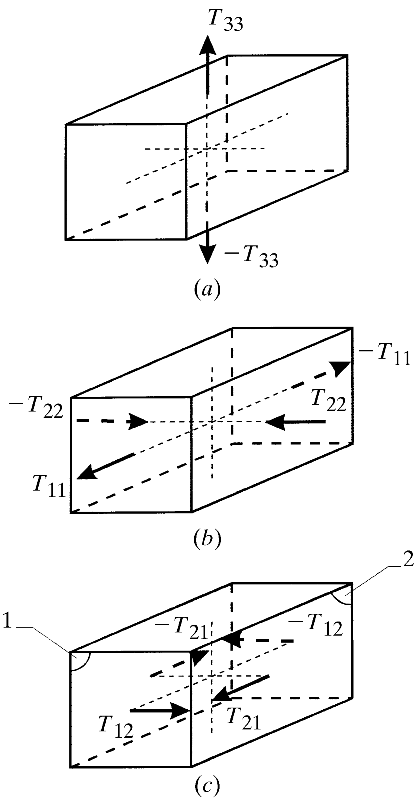

Fig. 1.3.2.5. Special forms of the stress tensor. (*a*) Uniaxial stress: the stress tensor has only one component, T_{33}; (*b*) pure shear stress: $T_{22} = -T_{11}$; (*c*) simple shear stress: $T_{21} = T_{12}$.

(apart from the volume forces mentioned earlier), the stress field inside the solid is such that at each point of the surface

$$T_{nj} = T_{ij}\alpha_i,$$

where the α_j's are the direction cosines of the normal to the surface at the point under consideration.

1.3.2.7. *Local properties of the stress tensor*

(i) *Normal stress and shearing stress*: let us consider a surface area element $d\sigma$ within the solid, the normal **n** to this element and the stress $\mathbf{T}_n$ that is applied to it (Fig. 1.3.2.6).

The *normal stress*, $\boldsymbol{\nu}$, is, by definition, the component of $\mathbf{T}_n$ on **n**,

$$\boldsymbol{\nu} = \mathbf{n}(\mathbf{T}_n \cdot \mathbf{n})$$

and the *shearing stress*, $\boldsymbol{\tau}$, is the projection of $\mathbf{T}_n$ on the surface area element,

$$\boldsymbol{\tau} = \mathbf{n} \wedge (\mathbf{T}_n \wedge \mathbf{n}) = \mathbf{T}_n - \boldsymbol{\nu}.$$

(ii) *The stress quadric*: let us consider the bilinear form attached to the stress tensor:

$$f(\mathbf{y}) = T_{ij}y_iy_j.$$

The quadric represented by

$$f(\mathbf{y}) = \varepsilon$$

is called the stress quadric, where $\varepsilon = \pm 1$. It may be an ellipsoid or a hyperboloid. Referred to the principal axes, and using Voigt's notation, its equation is

$$y_i^2 T_i = \varepsilon.$$

To every direction **n** of the medium, let us associate the radius vector **y** of the quadric (Fig. 1.3.2.7) through the relation

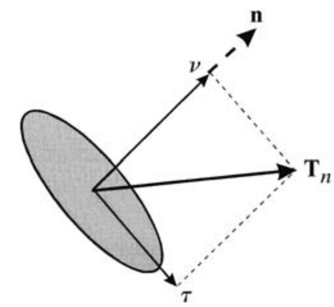

Fig. 1.3.2.6. Normal ($\boldsymbol{\nu}$) and shearing ($\boldsymbol{\tau}$) stress.

$$\mathbf{n} = k\mathbf{y}.$$

The stress applied to a small surface element $d\sigma$ normal to **n**, $\mathbf{T}_n$, is

$$\mathbf{T}_n = k\nabla(f)$$

and the normal stress, ν, is

$$\nu = \alpha_i T_i = 1/y^2,$$

where the α_i's are the direction cosines of **n**.

(iii) *Principal normal stresses*: the stress tensor is symmetrical and has therefore real eigenvectors. If we represent the tensor with reference to a system of axes parallel to its eigenvectors, it is put in the form

$$\begin{pmatrix} T_1 & 0 & 0 \\ 0 & T_2 & 0 \\ 0 & 0 & T_3 \end{pmatrix}.$$

T_1, T_2 and T_3 are the principal normal stresses. The mean normal stress, T, is defined by the relation

$$T = (T_1 + T_2 + T_3)/3$$

and is an invariant of the stress tensor.

1.3.2.8. *Energy density in a deformed medium*

Consider a medium that is subjected to a stress field T_{ij}. It has sustained a deformation indicated by the deformation tensor S. During this deformation, the forces of contact have performed work and the medium has accumulated a certain elastic energy W. The knowledge of the energy density thus acquired is useful for studying the properties of the elastic constants. Let the medium deform from the deformation S_{ij} to the deformation $S_{ij} + \delta S_{ij}$ under the influence of the stress field and let us evaluate the work of each component of the effort. Consider a small elementary rectangular parallelepiped of sides $2\Delta x_1$, $2\Delta x_2$, $2\Delta x_3$ (Fig. 1.3.2.8). We shall limit our calculation to the components T_{11} and T_{12}, which are applied to the faces 1 and 1′, respectively.

In the deformation δS, the point P goes to the point P', defined by

$$\mathbf{PP}' = \mathbf{u}(\mathbf{r}).$$

A neighbouring point Q goes to Q' such that (Fig. 1.3.1.1)

$$\mathbf{PQ} = \Delta\mathbf{r}; \quad \mathbf{P}'\mathbf{Q}' = \delta\mathbf{r}'.$$

The coordinates of $\delta\mathbf{r}'$ are given by

$$\delta x_i' = \delta\Delta x_i + \delta S_{ij}\delta x_j.$$

Of sole importance is the relative displacement of Q with respect to P and the displacement that must be taken into account in calculating the forces applied at Q. The coordinates of the relative displacement are