

## 1.3. ELASTIC PROPERTIES

an extrinsic property. For example, the introduction of carbon into iron modifies considerably the extent of the elastic region.

The extents of the elastic and plastic regions vary appreciably from one material to another. Fragile materials, for instance, have a much reduced plastic region, with a clear break.

## 1.3.3.2. Elastic constants

## 1.3.3.2.1. Definition

Young's modulus is not sufficient to describe the deformation of the bar: its diameter is reduced, in effect, during the elongation. One other coefficient, at least, is therefore necessary. In a general way, let us consider the deformation of a continuous anisotropic medium under the action of a field of applied stresses. We will generalize Hooke's law by writing that at each point there is a linear relation between the components  $T_{ij}$  of the stress tensor and the components  $S_{ij}$  of the strain tensor:

$$\begin{aligned} S_{ij} &= s_{ijkl} T_{kl} \\ T_{ij} &= c_{ijkl} S_{kl}. \end{aligned} \quad (1.3.3.2)$$

The quantities  $s_{ijkl}$  and  $c_{ijkl}$  are characteristic of the elastic properties of the medium if it is homogeneous and are independent of the point under consideration. Their tensorial nature can be shown using the demonstration illustrated in Section 1.1.3.4. Let us take the contracted product of the two sides of each of the two equations of (1.3.3.2) by the components  $x_i$  and  $y_j$  of any two vectors,  $\mathbf{x}$  and  $\mathbf{y}$ :

$$\begin{aligned} S_{ij} x_i y_j &= s_{ijkl} T_{kl} x_i y_j \\ T_{ij} x_i y_j &= c_{ijkl} S_{kl} x_i y_j. \end{aligned}$$

The left-hand sides are bilinear forms since  $S_{ij}$  and  $T_{ij}$  are second-rank tensors and the right-hand sides are quadrilinear forms, which shows that  $s_{ijkl}$  and  $c_{ijkl}$  are the components of fourth-rank tensors, the tensor of elastic *compliances* (or moduli) and the tensor of elastic *stiffnesses* (or coefficients), respectively. The number of their components is equal to 81.

Equations (1.3.3.2) are Taylor expansions limited to the first term. The higher terms involve sixth-rank tensors,  $s_{ijklmn}$  and  $c_{ijklmn}$ , with  $3^6 = 729$  coefficients, called third-order elastic compliances and stiffnesses and eighth-rank tensors with  $3^8 = 6561$  coefficients, called fourth-order elastic compliances and stiffnesses. They will be defined in Section 1.3.6.4. Tables for third-order elastic constants are given in Fumi (1951, 1952, 1987). The accompanying software to this volume enables these tables to be derived for any point group.

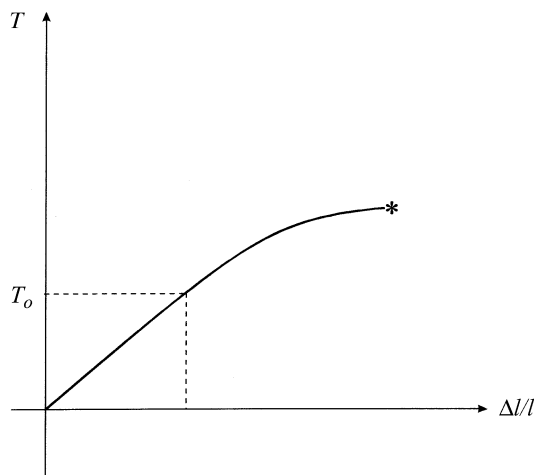


Fig. 1.3.3.2. Schematic stress–strain curve.  $T$ : stress;  $T_0$ : elastic limit;  $\Delta l/l$ : elongation; the asterisk symbolizes the rupture.

## 1.3.3.2.2. Matrix notation – reduction of the number of independent components

It is convenient to write the relations (1.3.3.2) in matrix form by associating with the stress and strain tensors column matrices  $1 \times 9$  and with the tensors of the elastic stiffnesses,  $c$ , and of the elastic compliances,  $s$ , square matrices  $9 \times 9$  (Section 1.1.4.10.4); these two  $9 \times 9$  matrices are inverse to one another. The number of independent components of the fourth-rank elastic tensors can be reduced by three types of consideration:

(i) *Intrinsic symmetry*: it was shown in Section 1.1.1.4 that tensors representing principal properties are symmetric. This is the case of the elastic tensors and can be shown directly using expression (1.3.2.7) of the energy stored per unit volume in the medium when we allow it to deform from the state  $S_{ij}$  to the state  $S_{ij} + \delta S_{ij}$  under the action of the stress  $T_{ij}$ :

$$\delta W = T_{ij} \delta S_{ij}.$$

Applying relation (1.3.3.1), we get

$$\partial W / \partial S_{ij} = c_{ijkl} S_{kl}. \quad (1.3.3.3)$$

Hence, one has by further differentiation

$$\partial^2 W / (\partial S_{ij} \partial S_{kl}) = c_{ijkl}.$$

Nothing is changed by interchanging the role of the pairs of dummy indices  $ij$  and  $lk$ :

$$\partial^2 W / (\partial S_{kl} \partial S_{ij}) = c_{klij}.$$

Since the energy is a state function with a perfect differential, one can interchange the order of the differentiations: the members on the left-hand sides of these two equations are therefore equal; one then deduces

$$c_{ijkl} = c_{klij}. \quad (1.3.3.4)$$

The tensor of elastic stiffnesses (and also the tensor of elastic compliances) is thus symmetrical. As shown in Section 1.1.4.5.2.2, the number of their independent components is therefore reduced to 45.

(ii) *Symmetry of the strain and stress tensors*: the strain tensor  $S_{ij}$  is symmetric by definition (Section 1.3.1.3.1) because rotations are not taken into account and the stress tensor  $T_{ij}$  is symmetric (Section 1.3.2.4) because body torques are neglected. For this reason, summation (1.3.3.2),  $S_{ij} = s_{ijkl} T_{kl}$ , can be factorized [equation (1.1.4.11)]:

$$S_{ij} = \sum_l s_{ijll} T_{ll} + \sum_{k \neq l} (s_{ijkl} + s_{ijlk}) T_{kl}.$$

This shows that the number of independent components of tensor  $s_{ijkl}$  is reduced. This effect of the symmetry of the strain and stress tensors was discussed systematically in Section 1.1.4.10.4. It was shown that

$$\begin{aligned} s_{ijkl} &= s_{ijlk} = s_{jikl} = s_{jilk} \\ c_{ijkl} &= c_{ijlk} = c_{jikl} = c_{jilk} \end{aligned} \quad (1.3.3.5)$$

and that the number of independent elastic compliances or stiffnesses is reduced to 21. They are replaced by two-index coefficients constituting  $6 \times 6$  matrices according to Voigt's notation: