

1.3. ELASTIC PROPERTIES

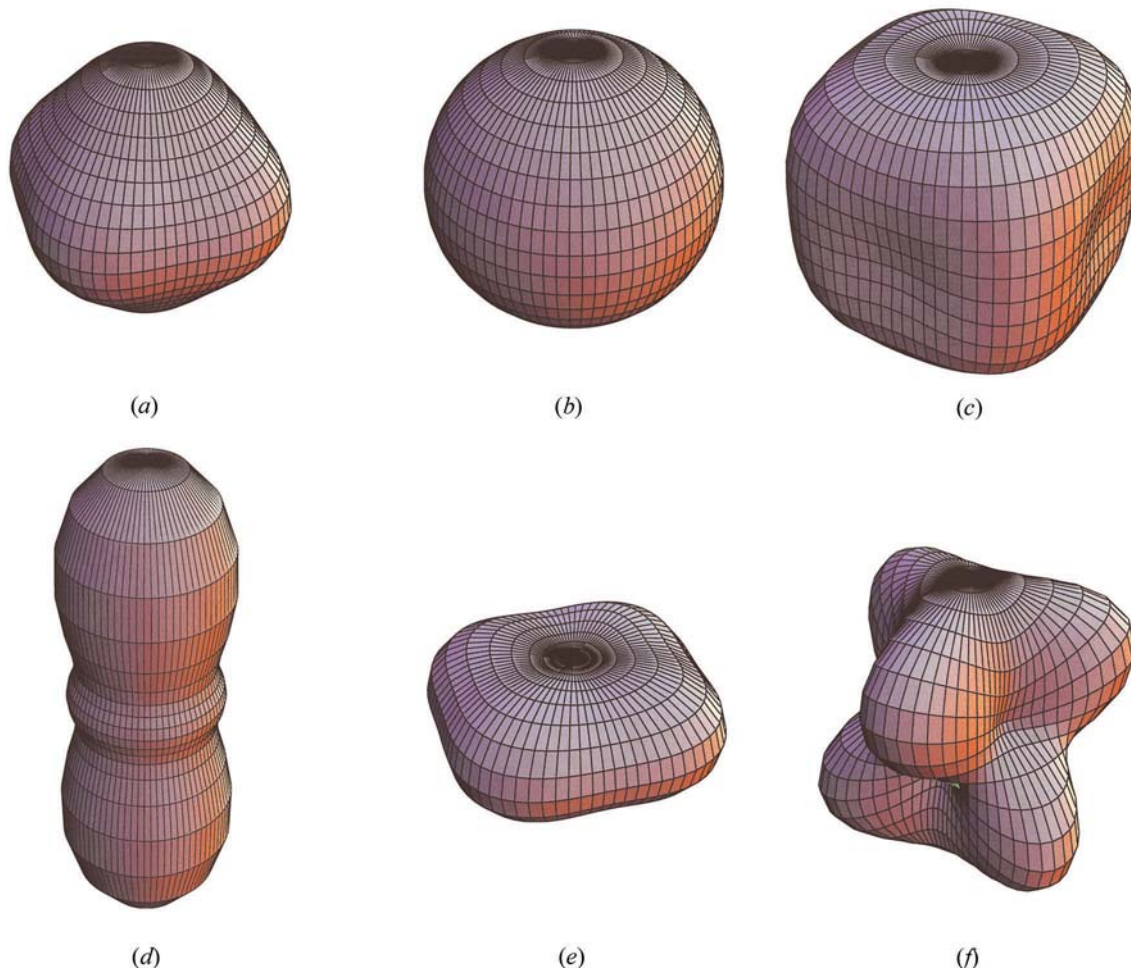


Fig. 1.3.3.4. Representation surface of the inverse of Young's modulus. (a) NaCl, cubic, anisotropy factor > 1; (b) W, cubic, anisotropy factor = 1; (c) Al, cubic, anisotropy factor < 1; (d) Zn, hexagonal; (e) Sn, tetragonal; (f) calcite, trigonal.

If one introduces the Lamé constants,

$$\begin{aligned} \mu &= (1/2)(c_{11} - c_{12}) = c_{44} \\ \lambda &= c_{12}, \end{aligned}$$

the equations may be written in the form often used in mechanics:

$$\begin{aligned} T_1 &= 2\mu S_1 + \lambda(S_1 + S_2 + S_3) \\ T_2 &= 2\mu S_2 + \lambda(S_1 + S_2 + S_3) \\ T_3 &= 2\mu S_3 + \lambda(S_1 + S_2 + S_3). \end{aligned} \quad (1.3.3.16)$$

Two coefficients suffice to define the elastic properties of an isotropic material, s_{11} and s_{12} , c_{11} and c_{12} , μ and λ , μ and ν , etc. Table 1.3.3.3 gives the relations between the more common elastic coefficients.

1.3.3.6. Equilibrium conditions of elasticity for isotropic media

We saw in Section 1.3.2.3 that the condition of equilibrium is

$$\partial T_{ij} / \partial x_i + \rho F_j = 0.$$

If we use the relations of elasticity, equation (1.3.3.2), this condition can be rewritten as a condition on the components of the strain tensor:

$$c_{ijkl} \frac{\partial S_{kl}}{\partial x_j} + \rho F_i = 0.$$

Recalling that

$$S_{kl} = \frac{1}{2} \left[\frac{\partial u_k}{\partial x_l} + \frac{\partial u_l}{\partial x_k} \right],$$

the condition becomes a condition on the displacement vector, $\mathbf{u}(\mathbf{r})$:

$$c_{ijkl} \frac{\partial^2}{\partial x_i \partial x_j} + \rho F_i = 0.$$

In an isotropic orthonormal medium, this equation, projected on the axis $0x_1$, can be written with the aid of relations (1.3.3.5) and (1.3.3.9):

$$\begin{aligned} &c_{11} \frac{\partial^2 u_1}{(\partial x_1)^2} + c_{12} \left[\frac{\partial^2 u_2}{\partial x_1 \partial x_2} + \frac{\partial^2 u_3}{\partial x_1 \partial x_3} \right] \\ &+ \frac{1}{2}(c_{11} - c_{12}) \left[\frac{\partial^2 u_1}{(\partial x_2)^2} + \frac{\partial^2 u_3}{\partial x_1 \partial x_3} + \frac{\partial^2 u_1}{(\partial x_3)^2} \right] + \rho F_1 \\ &= 0. \end{aligned}$$

This equation can finally be rearranged in one of the three following forms with the aid of Table 1.3.3.3.

$$\begin{aligned} &\frac{1}{2}(c_{11} - c_{12})\Delta \mathbf{u} + \frac{1}{2}(c_{11} + c_{12})\nabla(\nabla \mathbf{u}) + \rho \mathbf{F} = 0 \\ &\mu \Delta \mathbf{u} + (\mu + \lambda)\nabla(\nabla \mathbf{u}) + \rho \mathbf{F} = 0 \\ &\mu \left[\Delta \mathbf{u} + \frac{1}{1 - 2\nu} \nabla(\nabla \mathbf{u}) \right] + \rho \mathbf{F} = 0. \end{aligned} \quad (1.3.3.17)$$