International Tables for Crystallography (2006). Vol. D, References, pp. 97–98.

Table 1.3.7.1. *Relationships between ρW^2, its pressure derivatives and the second- and third-order elastic constants*

Propagation	Polarization	$(\bar{\rho}_0 W^2)_0$	$\partial(\bar{\rho}_0 W^2)_0/\partial p$
[100]	[100]	$\bar{c}_{11}$	$-1-(2\bar{c}_{11}+\Gamma_{1111})/3\bar{\kappa}$
[100]	[010]	$\bar{c}_{44}$	$-1-(2\bar{c}_{44}+\Gamma_{2323})/3\bar{\kappa}$
[110]	[110]	$(\bar{c}_{11}+\bar{c}_{12}+2\bar{c}_{44})/2$	$-1-(\bar{c}_{11}+\bar{c}_{12}+2\bar{c}_{44}+0.5[\Gamma_{1111}+\Gamma_{1122}+\Gamma_{2323}])/3\bar{\kappa}$
[110]	[1$\bar{1}$0]	$(\bar{c}_{11}-\bar{c}_{12}+\bar{c}_{44})/3$	$-1-(\bar{c}_{11}-\bar{c}_{11}+0.5[\Gamma_{1111}-\Gamma_{1122}])/3\bar{\kappa}$
[110]	[001]	$\bar{c}_{44}$	$-1-(2\bar{c}_{44}+\Gamma_{2323})/3\bar{\kappa}$
[111]	[111]	$(\bar{c}_{11}+2\bar{c}_{12}+4\bar{c}_{44})/3$	$-1-(2\bar{c}_{11}+4\bar{c}_{12}+8\bar{c}_{14}+[\Gamma_{1111}+2\Gamma_{1122}+4\Gamma_{2323}])/9\bar{\kappa}$
[111]	[1$\bar{1}$0]	$(\bar{c}_{11}-\bar{c}_{12}+\bar{c}_{44})/3$	$-1-(2\bar{c}_{11}-2\bar{c}_{12}+2\bar{c}_{14}+[\Gamma_{1111}-\Gamma_{1122}+\Gamma_{2323}])/9\bar{\kappa}$

or

$$\rho_0 \omega^2 A_j = \Delta_{jk} A_k$$

with $\Delta_{jk} = D_{ijkl} k_j k_l$.

The quantities $\rho_0 \omega^2 A_j$ and A are, respectively, the *eigenvalues* and *eigenvectors of the matrix* Δ_{jk}. Since Δ_{jk} is a real symmetric matrix, the eigenvalues are real and the eigenvectors are orthogonal.

1.3.7.6. *Experimental determination of third- and higher-order elastic constants*

The main experimental procedures for determining the third- and higher-order elastic constants are based on the measurement of stress derivatives of ultrasonic velocities and on harmonic generation experiments. Hydrostatic pressure, which can be accurately measured, has been widely used; however, the measurement of ultrasonic velocities in a solid under hydrostatic pressure cannot lead to the whole set of third-order elastic constants, so uniaxial stress measurements or harmonic generation experiments are then necessary.

In order to interpret wave-propagation measurements in stressed crystals, Thurston (1964) and Brugger (1964) introduced the concept of natural velocity with the following comments:

'According to equation of motion, the wave front is a material plane which has unit normal **k** in the natural state; a wave front moves from the plane $\mathbf{k}\cdot\mathbf{a}=\mathbf{0}$ to the plane $\mathbf{k}\cdot\mathbf{a}=\mathbf{L_0}$ in the time L_0/W. Thus W, *the natural velocity*, is the wave speed referred to natural dimensions for propagation normal to a plane of natural normal **k**.

In a typical ultrasonic experiment, plane waves are reflected between opposite parallel faces of a specimen, the wave fronts being parallel to these faces. One ordinarily measures a repetition frequency F, which is the inverse of the time required for a round trip between the opposite faces.'

Hence

$$W = 2L_0F.$$

In most experiments, the third-order elastic constants and higher-order elastic constants are deduced from the stress derivatives of $\bar{\rho}_0 W^2$. For instance, Table 1.3.7.1 gives the expressions for $(\bar{\rho}_0 W^2)_0$ and $\partial(\bar{\rho}_0 W^2)_0/\partial p$ for a cubic crystal. These quantities refer to the natural state free of stress. In this table, p denotes the hydrostatic pressure and the Γ_{ijkl}'s are the following linear combinations of third-order elastic constants:

$$\Gamma_{1111} = \bar{c}_{111} + 2\bar{c}_{111}$$
$$\Gamma_{1122} = 2\bar{c}_{112} + \bar{c}_{123}$$
$$\Gamma_{2323} = \bar{c}_{144} + 2\bar{c}_{166}.$$

1.3.8. Glossary

$\mathbf{e}_i$	covariant basis vector
A^T	transpose of matrix A
u_i	components of the displacement vector
S_{ij}	components of the strain tensor
S_α	components of the strain Voigt matrix
T_{ij}	components of the stress tensor
T_α	components of the stress Voigt matrix
p	pressure
$\boldsymbol{\nu}$	normal stress
$\boldsymbol{\tau}$	shear stress
s_{ijkl}	second-order elastic compliances
$s_{\alpha\beta}$	reduced second-order elastic compliances
$(s_{ijkl})^\sigma$	adiabatic second-order elastic compliances
s_{ijklmn}	third-order elastic compliances
c_{ijkl}	second-order elastic stiffnesses
$(c_{ijkl})^\sigma$	adiabatic second-order elastic stiffnesses
$c_{\alpha\beta}$	reduced second-order elastic stiffnesses
c_{ijklmn}	third-order elastic stiffnesses
ν	Poisson's ratio
E	Young's modulus
κ	bulk modulus (volume compressibility)
λ, μ	Lamé constants
Θ	temperature
c^S	specific heat at constant strain
ρ	volumic mass
Θ_D	Debye temperature
k_B	Boltzmann constant
U	internal energy
F	free energy

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