1.3. Elastic properties

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1.3.1. Strain tensor

1.3.1.1. *Introduction, the notion of strain field*

Let us consider a medium that undergoes a deformation. This means that the various points of the medium are displaced with respect to one another. Geometrical transformations of the medium that reduce to a translation of the medium as a whole will therefore not be considered. We may then suppose that there is an invariant point, O, whose position one can always return to by a suitable translation. A point P, with position vector $\mathbf{OP} = \mathbf{r}$, is displaced to the neighbouring point P' by the deformation defined by

$$\mathbf{PP}' = \mathbf{u}(\mathbf{r}).$$

The displacement vector $\mathbf{u}(\mathbf{r})$ constitutes a vector field. It is not a uniform field, unless the deformation reduces to a translation of the whole body, which is incompatible with the hypothesis that the medium undergoes a deformation. Let Q be a point that is near P before the deformation (Fig. 1.3.1.1). Then one can write

$$\mathrm{d}\mathbf{r} = \mathbf{PQ}; \quad \mathbf{r} + \mathrm{d}\mathbf{r} = \mathbf{OQ}.$$

After the deformation, Q is displaced to Q' defined by

$$\mathbf{QQ}' = \mathbf{u}(\mathbf{r} + \mathrm{d}\mathbf{r}).$$

In a deformation, it is more interesting in general to analyse the local, or relative, deformation than the absolute displacement. The relative displacement is given by comparing the vectors $\mathbf{P}'\mathbf{Q}' = \mathrm{d}\mathbf{r}'$ and $\mathbf{PQ}$. Thus, one has

$$\mathbf{P}'\mathbf{Q}' = \mathbf{P}'\mathbf{P} + \mathbf{PQ} + \mathbf{QQ}'.$$

Let us set

$$\left.\begin{aligned} \mathrm{d}\mathbf{r}' &= \mathrm{d}\mathbf{r} + \mathbf{u}(\mathbf{r} + \mathrm{d}\mathbf{r}) - \mathbf{u}(\mathbf{r}) \\ \mathrm{d}\mathbf{u} &= \mathbf{u}(\mathbf{r} + \mathrm{d}\mathbf{r}) - \mathbf{u}(\mathbf{r}) = \mathrm{d}\mathbf{r}' - \mathrm{d}\mathbf{r}. \end{aligned}\right\} \qquad (1.3.1.1)$$

Replacing $\mathbf{u}(\mathbf{r} + \mathrm{d}\mathbf{r})$ by its expansion up to the first term gives

$$\left.\begin{aligned} \mathrm{d}u_i &= \frac{\partial u_i}{\partial x_j}\,\mathrm{d}x_j \\ \mathrm{d}x'_i &= \mathrm{d}x_i + \frac{\partial u_i}{\partial x_j}\,\mathrm{d}x_j. \end{aligned}\right\} \qquad (1.3.1.2)$$

If we assume the Einstein convention (see Section 1.1.2.1), there is summation over j in (1.3.1.2) and (1.3.1.3). We shall further assume orthonormal coordinates throughout Chapter 1.3; variance is therefore not apparent and the positions of the indices have no meaning; the Einstein convention then only assumes repetition of a dummy index. The elements $\mathrm{d}x_i$ and $\mathrm{d}x'_i$ are the components of $\mathrm{d}\mathbf{r}$ and $\mathrm{d}\mathbf{r}'$, respectively. Let us put

$$M_{ij} = \partial u_i/\partial x_j; \;\; B_{ij} = M_{ij} + \delta_{ij},$$

where δ_{ij} represents the Kronecker symbol; the δ_{ij}'s are the components of matrix unity, I. The expressions (1.3.1.2) can also be written using matrices M and B:

$$\left.\begin{aligned} \mathrm{d}u_i &= M_{ij}\mathrm{d}x_j \\ \mathrm{d}x'_i &= B_{ij}\mathrm{d}x_j. \end{aligned}\right\} \qquad (1.3.1.3)$$

The components of the tensor M_{ij} are nonzero, unless, as mentioned earlier, the deformation reduces to a simple translation. Two cases in particular are of interest and will be discussed in turn:

(i) The components M_{ij} are constants. In this case, the deformation is homogeneous.

(ii) The components M_{ij} are variables but are small compared with unity. This is the practical case to which we shall limit ourselves in considering an inhomogeneous deformation.

1.3.1.2. *Homogeneous deformation*

If the components M_{ij} are constants, equations (1.3.1.3) can be integrated directly. They become, to a translation,

$$\left.\begin{aligned} u_i &= M_{ij}x_j \\ x'_i &= B_{ij}x_j. \end{aligned}\right\} \qquad (1.3.1.4)$$

1.3.1.2.1. *Fundamental property of the homogeneous deformation*

The fundamental property of the homogeneous deformation results from the fact that equations (1.3.1.4) are linear: a plane before the deformation remains a plane afterwards, a crystal lattice remains a lattice. Thermal expansion is a homogeneous deformation (see Chapter 1.4).

1.3.1.2.2. *Spontaneous strain*

Some crystals present a twin microstructure that is seen to change when the crystals are gently squeezed. At rest, the domains can have one of two different possible orientations and the influence of an applied stress is to switch them from one orientation to the other. If one measures the shape of the crystal lattice (the strain of the lattice) as a function of the applied stress, one obtains an elastic hysteresis loop analogous to the magnetic or electric hysteresis loops observed in ferromagnetic or ferroelectric crystals. For this reason, these materials are called *ferroelastic* (see Chapters 3.1 to 3.3 and Salje, 1990). The strain associated with one of the two possible shapes of the crystal when no stress is applied is called the macroscopic *spontaneous strain*.

1.3.1.2.3. *Cubic dilatation*

Let $\mathbf{e}_i$ be the basis vectors before deformation. On account of the deformation, they are transformed into the three vectors

$$\mathbf{e}'_i = B_{ij}\mathbf{e}_j.$$

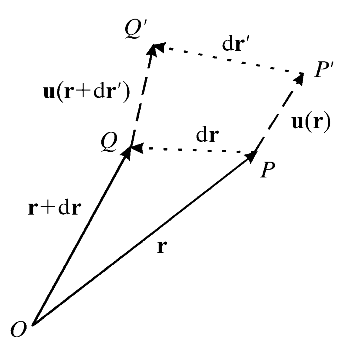

Fig. 1.3.1.1. Displacement vector, $\mathbf{u}(\mathbf{r})$.