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1.3. Elastic properties

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1.3.1. Strain tensor

1.3.1.1. *Introduction, the notion of strain field*

Let us consider a medium that undergoes a deformation. This means that the various points of the medium are displaced with respect to one another. Geometrical transformations of the medium that reduce to a translation of the medium as a whole will therefore not be considered. We may then suppose that there is an invariant point, O, whose position one can always return to by a suitable translation. A point P, with position vector $\mathbf{OP} = \mathbf{r}$, is displaced to the neighbouring point P' by the deformation defined by

$$\mathbf{PP}' = \mathbf{u}(\mathbf{r}).$$

The displacement vector $\mathbf{u}(\mathbf{r})$ constitutes a vector field. It is not a uniform field, unless the deformation reduces to a translation of the whole body, which is incompatible with the hypothesis that the medium undergoes a deformation. Let Q be a point that is near P before the deformation (Fig. 1.3.1.1). Then one can write

$$\mathrm{d}\mathbf{r} = \mathbf{PQ}; \quad \mathbf{r} + \mathrm{d}\mathbf{r} = \mathbf{OQ}.$$

After the deformation, Q is displaced to Q' defined by

$$\mathbf{QQ}' = \mathbf{u}(\mathbf{r} + \mathrm{d}\mathbf{r}).$$

In a deformation, it is more interesting in general to analyse the local, or relative, deformation than the absolute displacement. The relative displacement is given by comparing the vectors $\mathbf{P}'\mathbf{Q}' = \mathrm{d}\mathbf{r}'$ and $\mathbf{PQ}$. Thus, one has

$$\mathbf{P}'\mathbf{Q}' = \mathbf{P}'\mathbf{P} + \mathbf{PQ} + \mathbf{QQ}'.$$

Let us set

$$\left.\begin{aligned} \mathrm{d}\mathbf{r}' &= \mathrm{d}\mathbf{r} + \mathbf{u}(\mathbf{r} + \mathrm{d}\mathbf{r}) - \mathbf{u}(\mathbf{r}) \\ \mathrm{d}\mathbf{u} &= \mathbf{u}(\mathbf{r} + \mathrm{d}\mathbf{r}) - \mathbf{u}(\mathbf{r}) = \mathrm{d}\mathbf{r}' - \mathrm{d}\mathbf{r}. \end{aligned}\right\} \quad (1.3.1.1)$$

Replacing $\mathbf{u}(\mathbf{r} + \mathrm{d}\mathbf{r})$ by its expansion up to the first term gives

$$\left.\begin{aligned} \mathrm{d}u_i &= \frac{\partial u_i}{\partial x_j}\,\mathrm{d}x_j \\ \mathrm{d}x'_i &= \mathrm{d}x_i + \frac{\partial u_i}{\partial x_j}\,\mathrm{d}x_j. \end{aligned}\right\} \quad (1.3.1.2)$$

If we assume the Einstein convention (see Section 1.1.2.1), there is summation over j in (1.3.1.2) and (1.3.1.3). We shall further assume orthonormal coordinates throughout Chapter 1.3; variance is therefore not apparent and the positions of the indices have no meaning; the Einstein convention then only assumes repetition of a dummy index. The elements $\mathrm{d}x_i$ and $\mathrm{d}x'_i$ are the components of $\mathrm{d}\mathbf{r}$ and $\mathrm{d}\mathbf{r}'$, respectively. Let us put

$$M_{ij} = \partial u_i/\partial x_j; \;\; B_{ij} = M_{ij} + \delta_{ij},$$

where δ_{ij} represents the Kronecker symbol; the δ_{ij}'s are the components of matrix unity, I. The expressions (1.3.1.2) can also be written using matrices M and B:

$$\left.\begin{aligned} \mathrm{d}u_i &= M_{ij}\mathrm{d}x_j \\ \mathrm{d}x'_i &= B_{ij}\mathrm{d}x_j. \end{aligned}\right\} \quad (1.3.1.3)$$

The components of the tensor M_{ij} are nonzero, unless, as mentioned earlier, the deformation reduces to a simple translation. Two cases in particular are of interest and will be discussed in turn:

(i) The components M_{ij} are constants. In this case, the deformation is homogeneous.

(ii) The components M_{ij} are variables but are small compared with unity. This is the practical case to which we shall limit ourselves in considering an inhomogeneous deformation.

1.3.1.2. *Homogeneous deformation*

If the components M_{ij} are constants, equations (1.3.1.3) can be integrated directly. They become, to a translation,

$$\left.\begin{aligned} u_i &= M_{ij}x_j \\ x'_i &= B_{ij}x_j. \end{aligned}\right\} \quad (1.3.1.4)$$

1.3.1.2.1. *Fundamental property of the homogeneous deformation*

The fundamental property of the homogeneous deformation results from the fact that equations (1.3.1.4) are linear: a plane before the deformation remains a plane afterwards, a crystal lattice remains a lattice. Thermal expansion is a homogeneous deformation (see Chapter 1.4).

1.3.1.2.2. *Spontaneous strain*

Some crystals present a twin microstructure that is seen to change when the crystals are gently squeezed. At rest, the domains can have one of two different possible orientations and the influence of an applied stress is to switch them from one orientation to the other. If one measures the shape of the crystal lattice (the strain of the lattice) as a function of the applied stress, one obtains an elastic hysteresis loop analogous to the magnetic or electric hysteresis loops observed in ferromagnetic or ferroelectric crystals. For this reason, these materials are called *ferroelastic* (see Chapters 3.1 to 3.3 and Salje, 1990). The strain associated with one of the two possible shapes of the crystal when no stress is applied is called the macroscopic *spontaneous strain*.

1.3.1.2.3. *Cubic dilatation*

Let $\mathbf{e}_i$ be the basis vectors before deformation. On account of the deformation, they are transformed into the three vectors

$$\mathbf{e}'_i = B_{ij}\mathbf{e}_j.$$

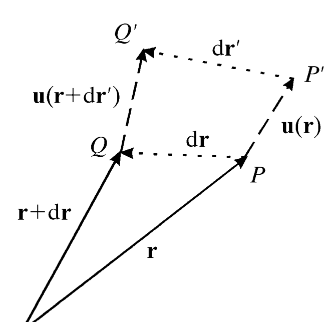

Fig. 1.3.1.1. Displacement vector, $\mathbf{u}(\mathbf{r})$.

The parallelepiped formed by these three vectors has a volume V' given by

$$V' = (\mathbf{e}'_1, \mathbf{e}'_2, \mathbf{e}'_3) = \Delta(B)(\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3) = \Delta(B)V,$$

where $\Delta(B)$ is the determinant associated with matrix B, V is the volume before deformation and

$$(\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3) = (\mathbf{e}_1 \wedge \mathbf{e}_2) \cdot \mathbf{e}_3$$

represents a triple scalar product.

The relative variation of the volume is

$$\frac{V' - V}{V} = \Delta(B) - 1. \qquad (1.3.1.5)$$

It is what one calls the *cubic dilatation*. $\Delta(B)$ gives directly the volume of the parallelepiped that is formed from the three vectors obtained in the deformation when starting from vectors forming an orthonormal base.

1.3.1.2.4. *Expression of any homogeneous deformation as the product of a pure rotation and a pure deformation*

(i) *Pure rotation*: It is isometric. The moduli of the vectors remain unchanged and one direction remains invariant, the axis of rotation. The matrix B is unitary:

$$BB^T = 1.$$

(ii) *Pure deformation*: This is a deformation in which three orthogonal directions remain invariant. It can be shown that B is a symmetric matrix:

$$B = B^T.$$

The three invariant directions are those of the eigenvectors of the matrix; it is known in effect that the eigenvectors of a symmetric matrix are real.

(iii) *Arbitrary deformation*: the matrix B, representing an arbitrary deformation, can always be put into the form of the product of a unitary matrix B_1, representing a pure rotation, and a symmetric matrix B_2, representing a pure deformation. Let us put

$$B = B_1 B_2$$

and consider the transpose matrix of B:

$$B^T = B_2^T B_1^T = B_2 (B_1)^{-1}.$$

The product $B^T B$ is equal to

$$B^T B = (B_2)^2.$$

This shows that we can determine B_2 and therefore B_1 from B.

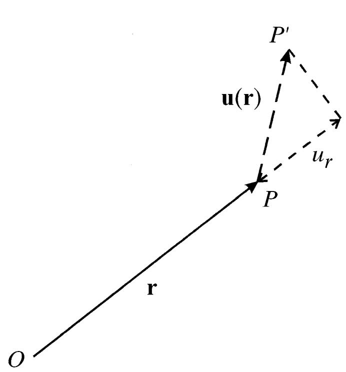

Fig. 1.3.1.2. Elongation, u_r/r.

1.3.1.2.5. *Quadric of elongations*

Let us project the displacement vector $\mathbf{u}(\mathbf{r})$ on the position vector $\mathbf{OP}$ (Fig. 1.3.1.2), and let u_r be this projection. The *elongation* is the quantity defined by

$$\frac{u_r}{r} = \frac{\mathbf{u} \cdot \mathbf{r}}{r^2} = \frac{M_{ij} x_i x_j}{r^2},$$

where x_1, x_2, x_3 are the components of $\mathbf{r}$. The elongation is the relative variation of the length of the vector $\mathbf{r}$ in the deformation. Let A and S be the antisymmetric and symmetric parts of M, respectively:

$$A = \frac{M - M^T}{2}; \quad S = \frac{M + M^T}{2}.$$

Only the symmetric part of M occurs in the expression of the elongation:

$$\frac{u_r}{r} = \frac{S_{ij} x_i x_j}{r^2}. \qquad (1.3.1.6)$$

The geometrical study of the elongation as a function of the direction of $\mathbf{r}$ is facilitated by introducing the quadric associated with M:

$$S_{ij} y_i y_j = \varepsilon, \qquad (1.3.1.7)$$

where ε is a constant. This quadric is called the *quadric of elongations*, Q. S is a symmetric matrix with three real orthogonal eigenvectors and three real eigenvalues, λ_1, λ_2, λ_3. If it is referred to these axes, equation (1.3.1.7) is reduced to

$$\lambda_1 (y_1)^2 \lambda_2 (y_2)^2 \lambda_3 (y_3)^2 = \varepsilon.$$

One can discuss the form of the quadric according to the sign of the eigenvalues λ_i:

(i) λ_1, λ_2, λ_3 have the same sign, and the sign of ε. The quadric is an ellipsoid (Fig. 1.3.1.3*a*). One chooses $\varepsilon = +1$ or $\varepsilon = -1$, depending on the sign of the eigenvalues.

(ii) λ_1, λ_2, λ_3 are of mixed signs: one of them is of opposite sign to the other two. One takes $\varepsilon = \pm 1$. The corresponding quadric is a hyperboloid whose asymptote is the cone

$$S_{ij} y_i y_j = 0.$$

According to the sign of ε, the hyperboloid will have one sheet outside the cone or two sheets inside the cone (Fig. 1.3.1.3*b*). If we wish to be able to consider any direction of the position vector $\mathbf{r}$ in space, it is necessary to take into account the two quadrics.

In order to follow the variations of the elongation u_r/r with the orientation of the position vector, one associates with $\mathbf{r}$ a vector $\mathbf{y}$, which is parallel to it and is defined by

$$\mathbf{y} = \mathbf{r}/k; \quad \mathbf{r} = k\mathbf{y},$$

where k is a constant. It can be seen that, in accordance with (1.3.1.6) and (1.3.1.7), the expression of the elongation in terms of y is

$$u_r/r = \varepsilon/y^2.$$

Thus, the elongation is inversely proportional to the square of the radius vector of the quadric of elongations parallel to $\mathbf{OP}$. In practice, it is necessary to look for the intersection p of the parallel to $\mathbf{OP}$ drawn from the centre O of the quadric of elongations (Fig. 1.3.1.3*a*):

(i) The eigenvalues all have the same sign; the quadric Q is an ellipsoid: the elongation has the same sign in all directions in space, positive for $\varepsilon = +1$ and negative for $\varepsilon = -1$.

(ii) The eigenvalues have different signs; two quadrics are to be taken into account: the hyperboloids corresponding, respectively, to $\varepsilon = \pm 1$. The sign of the elongation is different according to