International Tables for Crystallography (2006). Vol. D, Section 1.3.1.2.5, pp. 73–74.

The parallelepiped formed by these three vectors has a volume V' given by

$$V' = (\mathbf{e}'_1, \mathbf{e}'_2, \mathbf{e}'_3) = \Delta(B)(\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3) = \Delta(B)V,$$

where $\Delta(B)$ is the determinant associated with matrix B, V is the volume before deformation and

$$(\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3) = (\mathbf{e}_1 \wedge \mathbf{e}_2) \cdot \mathbf{e}_3$$

represents a triple scalar product.

The relative variation of the volume is

$$\frac{V' - V}{V} = \Delta(B) - 1. \qquad (1.3.1.5)$$

It is what one calls the *cubic dilatation.* $\Delta(B)$ gives directly the volume of the parallelepiped that is formed from the three vectors obtained in the deformation when starting from vectors forming an orthonormal base.

1.3.1.2.4. *Expression of any homogeneous deformation as the product of a pure rotation and a pure deformation*

(i) *Pure rotation*: It is isometric. The moduli of the vectors remain unchanged and one direction remains invariant, the axis of rotation. The matrix B is unitary:

$$BB^T = 1.$$

(ii) *Pure deformation*: This is a deformation in which three orthogonal directions remain invariant. It can be shown that B is a symmetric matrix:

$$B = B^T.$$

The three invariant directions are those of the eigenvectors of the matrix; it is known in effect that the eigenvectors of a symmetric matrix are real.

(iii) *Arbitrary deformation*: the matrix B, representing an arbitrary deformation, can always be put into the form of the product of a unitary matrix B_1, representing a pure rotation, and a symmetric matrix B_2, representing a pure deformation. Let us put

$$B = B_1 B_2$$

and consider the transpose matrix of B:

$$B^T = B_2^T B_1^T = B_2(B_1)^{-1}.$$

The product $B^T B$ is equal to

$$B^T B = (B_2)^2.$$

This shows that we can determine B_2 and therefore B_1 from B.

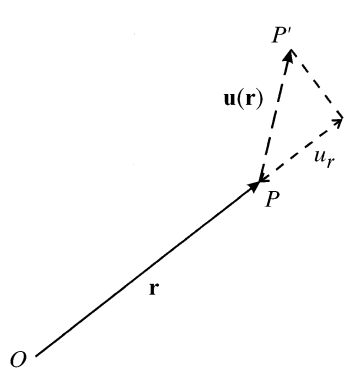

Fig. 1.3.1.2. Elongation, u_r/r.

1.3.1.2.5. *Quadric of elongations*

Let us project the displacement vector $\mathbf{u}(\mathbf{r})$ on the position vector $\mathbf{OP}$ (Fig. 1.3.1.2), and let u_r be this projection. The *elongation* is the quantity defined by

$$\frac{u_r}{r} = \frac{\mathbf{u} \cdot \mathbf{r}}{r^2} = \frac{M_{ij} x_i x_j}{r^2},$$

where x_1, x_2, x_3 are the components of $\mathbf{r}$. The elongation is the relative variation of the length of the vector $\mathbf{r}$ in the deformation. Let A and S be the antisymmetric and symmetric parts of M, respectively:

$$A = \frac{M - M^T}{2}; \quad S = \frac{M + M^T}{2}.$$

Only the symmetric part of M occurs in the expression of the elongation:

$$\frac{u_r}{r} = \frac{S_{ij} x_i x_j}{r^2}. \qquad (1.3.1.6)$$

The geometrical study of the elongation as a function of the direction of $\mathbf{r}$ is facilitated by introducing the quadric associated with M:

$$S_{ij} y_i y_j = \varepsilon, \qquad (1.3.1.7)$$

where ε is a constant. This quadric is called the *quadric of elongations*, Q. S is a symmetric matrix with three real orthogonal eigenvectors and three real eigenvalues, λ_1, λ_2, λ_3. If it is referred to these axes, equation (1.3.1.7) is reduced to

$$\lambda_1 (y_1)^2 \lambda_2 (y_2)^2 \lambda_3 (y_3)^2 = \varepsilon.$$

One can discuss the form of the quadric according to the sign of the eigenvalues λ_i:

(i) λ_1, λ_2, λ_3 have the same sign, and the sign of ε. The quadric is an ellipsoid (Fig. 1.3.1.3*a*). One chooses $\varepsilon = +1$ or $\varepsilon = -1$, depending on the sign of the eigenvalues.

(ii) λ_1, λ_2, λ_3 are of mixed signs: one of them is of opposite sign to the other two. One takes $\varepsilon = \pm 1$. The corresponding quadric is a hyperboloid whose asymptote is the cone

$$S_{ij} y_i y_j = 0.$$

According to the sign of ε, the hyperboloid will have one sheet outside the cone or two sheets inside the cone (Fig. 1.3.1.3*b*). If we wish to be able to consider any direction of the position vector $\mathbf{r}$ in space, it is necessary to take into account the two quadrics.

In order to follow the variations of the elongation u_r/r with the orientation of the position vector, one associates with $\mathbf{r}$ a vector $\mathbf{y}$, which is parallel to it and is defined by

$$\mathbf{y} = \mathbf{r}/k; \quad \mathbf{r} = k\mathbf{y},$$

where k is a constant. It can be seen that, in accordance with (1.3.1.6) and (1.3.1.7), the expression of the elongation in terms of y is

$$u_r/r = \varepsilon/y^2.$$

Thus, the elongation is inversely proportional to the square of the radius vector of the quadric of elongations parallel to $\mathbf{OP}$. In practice, it is necessary to look for the intersection p of the parallel to $\mathbf{OP}$ drawn from the centre O of the quadric of elongations (Fig. 1.3.1.3*a*):

(i) The eigenvalues all have the same sign; the quadric Q is an ellipsoid: the elongation has the same sign in all directions in space, positive for $\varepsilon = +1$ and negative for $\varepsilon = -1$.

(ii) The eigenvalues have different signs; two quadrics are to be taken into account: the hyperboloids corresponding, respectively, to $\varepsilon = \pm 1$. The sign of the elongation is different according to whether the direction under consideration is outside or inside the asymptotic cone and intersects one or the other of the two hyperboloids.

Equally, one can connect the displacement vector $\mathbf{u}(\mathbf{r})$ directly with the quadric Q. Using the bilinear form

$$f(\mathbf{y}) = M_{ij} y_i y_j,$$

the gradient of $f(\mathbf{y})$, $\boldsymbol{\nabla}(f)$, has as components

$$\partial f / \partial y^i = M_{ij} y_j = u_i.$$

One recognizes the components of the displacement vector $\mathbf{u}$, which is therefore parallel to the normal to the quadric Q at the extremity of the radius vector $\mathbf{Op}$ parallel to $\mathbf{r}$.

The directions of the principal axes of Q correspond to the extremal values of y, *i.e.* to the stationary values (maximal or minimal) of the elongation. These values are the *principal elongations*.

If the deformation is a pure rotation

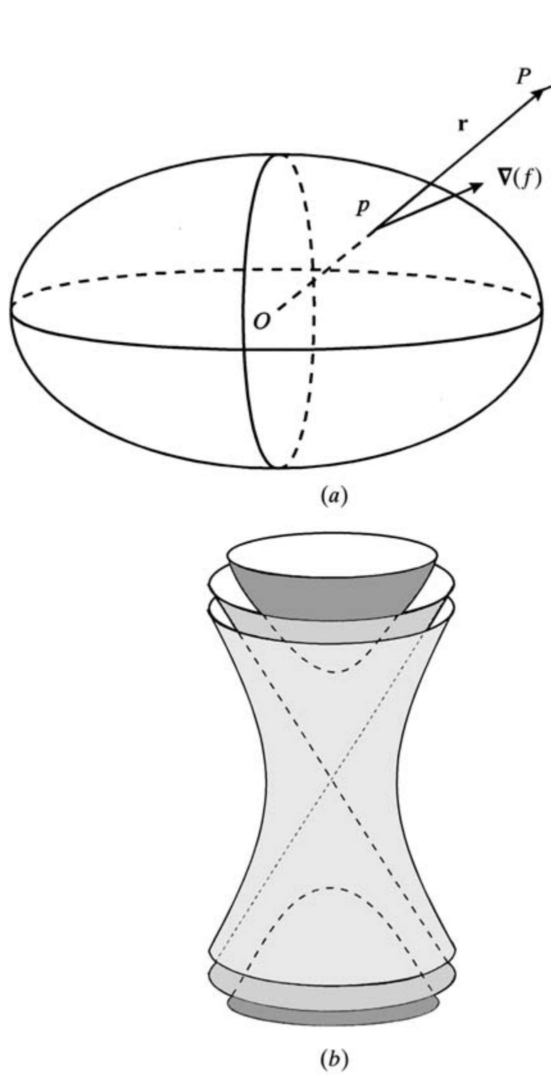

Fig. 1.3.1.3. Quadric of elongations. The displacement vector, $\mathbf{u}(\mathbf{r})$, at P in the deformed medium is parallel to the normal to the quadric at the intersection, p, of $\mathbf{OP}$ with the quadric. (a) The eigenvalues all have the same sign, the quadric is an ellipsoid. (b) The eigenvalues have mixed signs, the quadric is a hyperboloid with either one sheet (shaded in light grey) or two sheets (shaded in dark grey), depending on the sign of the constant ε [see equation (1.3.1.7)]; the cone asymptote is represented in medium grey. For a practical application, see Fig. 1.4.1.1.

$$B = \begin{pmatrix} \cos\theta & \sin\theta & 0 \\ -\sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{pmatrix},$$

$$M = \begin{pmatrix} \cos\theta - 1 & \sin\theta & 0 \\ -\sin\theta & \cos\theta - 1 & 0 \\ 0 & 0 & 0 \end{pmatrix}.$$

Hence we have

$$M_{ij} y_i y_j = (\cos\theta - 1)(y_1 - y_2) = \varepsilon.$$

The quadric Q is a cylinder of revolution having the axis of rotation as axis.

1.3.1.3. *Arbitrary but small deformations*

1.3.1.3.1. *Definition of the strain tensor*

If the deformation is small but arbitrary, *i.e.* if the products of two or more components of M_{ij} can be neglected with respect to unity, one can describe the deformation locally as a homogeneous asymptotic deformation. As was shown in Section 1.3.1.2.4, it can be put in the form of the product of a pure deformation corresponding to the symmetric part of M_{ij}, S_{ij}, and a pure rotation corresponding to the asymmetric part, A_{ij}:

$$\left.\begin{aligned} S_{ij} &= S_{ji} = \tfrac{1}{2}\left(\frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i}\right) \\ A_{ij} &= -A_{ji} = \tfrac{1}{2}\left(\frac{\partial u_i}{\partial x_j} - \frac{\partial u_j}{\partial x_i}\right). \end{aligned}\right\} \qquad (1.3.1.8)$$

Matrix B can be written

$$B = I + A + S,$$

where I is the matrix identity. As the coefficients $\partial u_i/\partial x_j$ of M_{ij} are small, one can neglect the product $A \times S$ and one has

$$B = (I + A)(I + S).$$

$(I + S)$ is a symmetric matrix that represents a pure deformation. $(I + A)$ is an antisymmetric unitary matrix and, since A is small,

$$(I + A)^{-1} = (I - A).$$

Thus, $(I + A)$ represents a rotation. The axis of rotation is parallel to the vector with coordinates

$$\left.\begin{aligned} \Omega_1 &= \tfrac{1}{2}\left(\frac{\partial u_3}{\partial x_2} + \frac{\partial u_2}{\partial x_3}\right) = A_{32} \\ \Omega_2 &= \tfrac{1}{2}\left(\frac{\partial u_1}{\partial x_3} + \frac{\partial u_3}{\partial x_1}\right) = A_{13} \\ \Omega_3 &= \tfrac{1}{2}\left(\frac{\partial u_2}{\partial x_1} + \frac{\partial u_1}{\partial x_2}\right) = A_{21}, \end{aligned}\right\}$$

which is an eigenvector of $(I + A)$. The magnitude of the rotation is equal to the modulus of this vector.

In general, one is only interested in the pure deformation, *i.e.* in the form of the deformed object. Thus, one only wishes to know the quantities $(I + S)$ and the symmetric part of M. It is this symmetric part that is called the deformation tensor or the strain tensor. It is very convenient for applications to use the simplified notation due to Voigt:

$$S_1 = \frac{\partial u_1}{\partial x_1}; \qquad S_2 = \frac{\partial u_2}{\partial x_2}; \qquad S_3 = \frac{\partial u_3}{\partial x_3};$$
$$S_4 = \frac{\partial u_3}{\partial x_2} + \frac{\partial u_2}{\partial x_3}; \quad S_5 = \frac{\partial u_3}{\partial x_1} + \frac{\partial u_1}{\partial x_3}; \quad S_6 = \frac{\partial u_2}{\partial x_1} + \frac{\partial u_1}{\partial x_2}.$$